Tensor principal component analysis via sum-of-squares proofs

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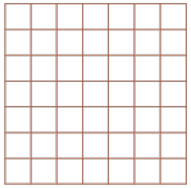
principal component analysis (PCA)

vanilla PCA

2-wise correlation data
= matrix



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- given noisy pairwise correlation data $A \in \mathbb{R}^{n \times n}$, find direction of maximum empirical variance

  maximize $\langle x, Ax \rangle$ over all unit vectors $x \in \mathbb{R}^n$

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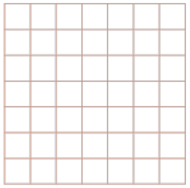

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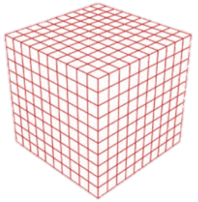
3-wise correlation data
= 3-tensor



variants of PCA

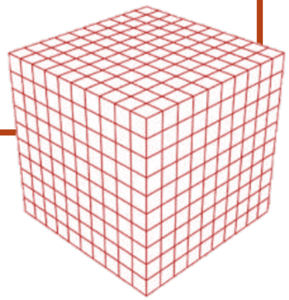
- restrict to sparse directions (SPARSE PCA) or exploit higher-order correlation data $A \in \mathbb{R}^{n \times n \times \cdots \times n}$ (TENSOR PCA)
- better statistical properties in important applications; huge body of works
- *but:* **computationally challenging** (NP-hard in worse case; unclear complexity in stochastic setting)

tensor principal component analysis

(k,τ)-stochastic model [Montanari-Richard]

given k-order tensor A as below, recover v (approximately)

$$A = \tau \cdot v^{\otimes k} + Z \in \mathbb{R}^{n^k} \text{ with } Z \sim N(0,1)^{\otimes k}$$



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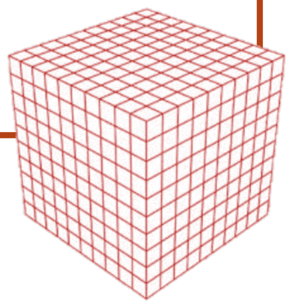

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noise: random tensor

signal: rank-1 tensor of unit vector $v \in \mathbb{R}^n$

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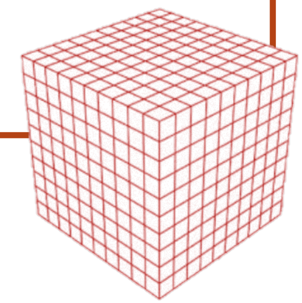

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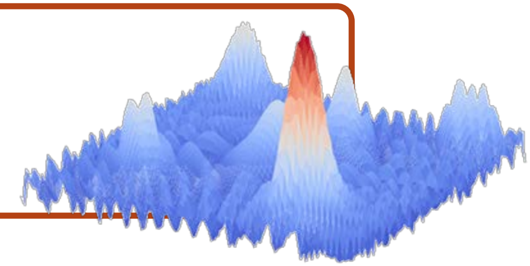

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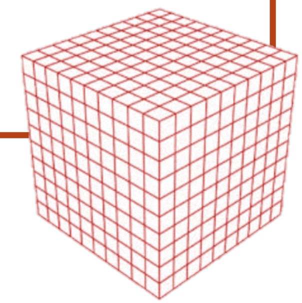

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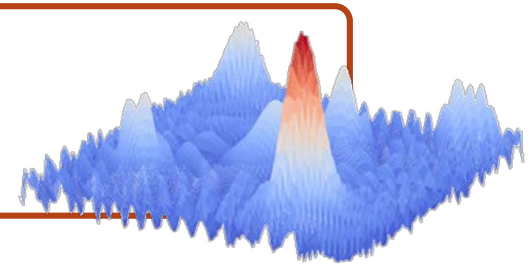

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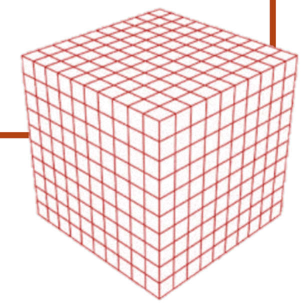

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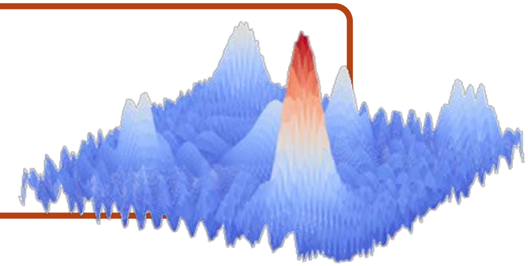

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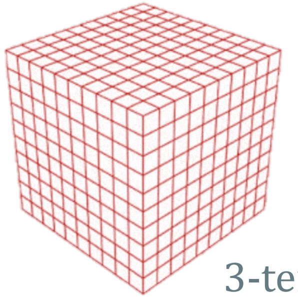

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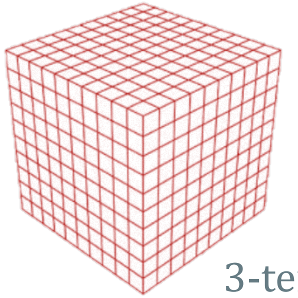

previous results [Montanari-Richard=MR]

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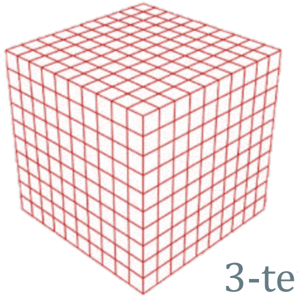

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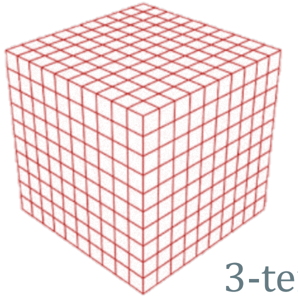

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theoretical guarantee: algorithm works as long as $\tau \geq \tilde{O}(n)$

empirical performance: algorithm works as long as $\tau \geq \tilde{O}(n^{3/4})$

tension: theoretical analysis of MR *tight* in many ways **but** empirical performance *should be predictive* for mathematical truth (average-case problem & large input sizes)

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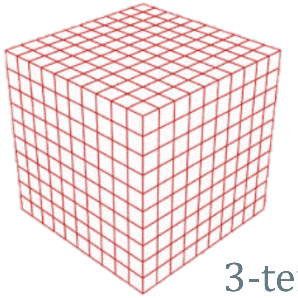
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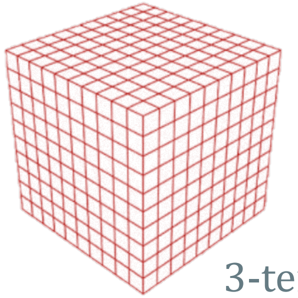

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techniques: sum-of-squares meta-algorithm & proof system;

powerful general approach to unsupervised learning

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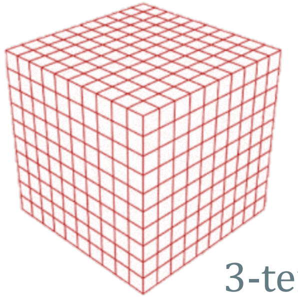

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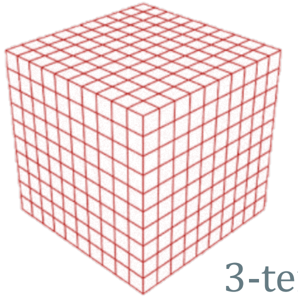

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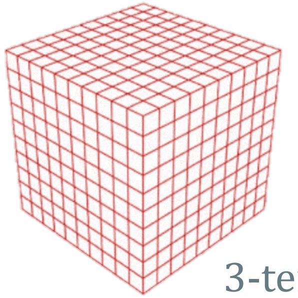

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lower bounds: rule out better recovery guarantees by algorithms based on broad set of techniques (deg-4 sum-of-squares proof system)

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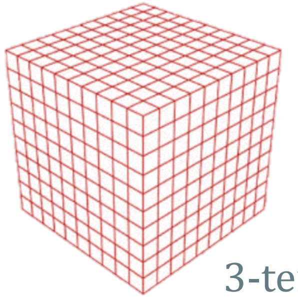

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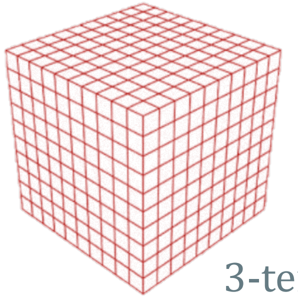

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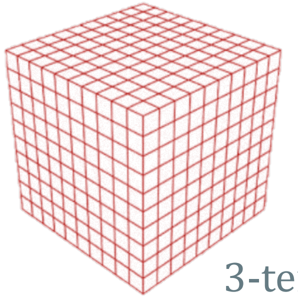

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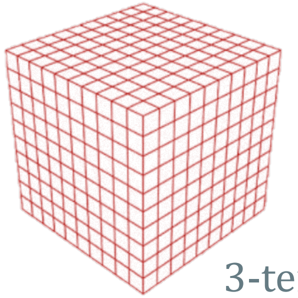

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our explanation *(for variant of MR relaxation)*
second-order effect in opt. value of relaxation drives recovery

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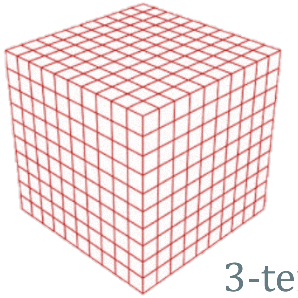
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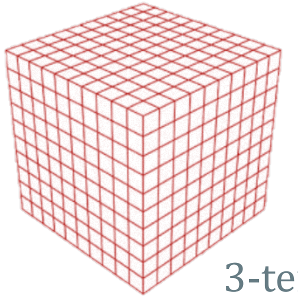

warm-up: upper bounds for homogeneous n-var. deg.-4 polynomial $p(x)$

consider affine linear subspace $H_{p(x)}$ of *matrix representations* of $p(x)$

$$H_{p(x)} \stackrel{\text{def}}{=} \left\{ P \in \mathbb{R}^{n^2 \times n^2} \,\middle|\, p(x) = \langle x^{\otimes 2}, P x^{\otimes 2} \rangle \right\}$$

$\lambda_{\max}(P) = \max_{\|y\|=1} \langle y, Py \rangle$

then, $\max_{\|x\|=1} p(x) \le \lambda_{\max}(P)$ for every $P \in H_{p(x)}$

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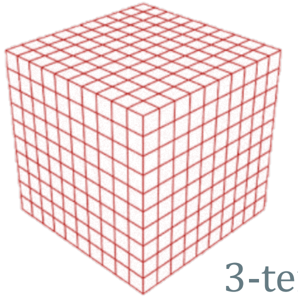

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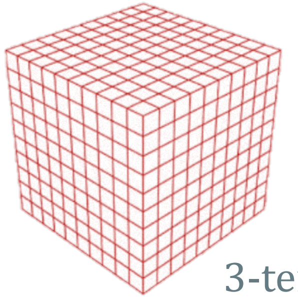
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deg-d sum-of-squares upper bounds for general polynomial $p(x)$

find *best upper bound* $\lambda_{\max}(P)$ with

$$P \in \bigcup_{\deg q(x) \le d-2} H_{p(x) + q(x) \cdot (\|x\|^2 - 1)}$$

different polynomials but same function as $p(x)$ on unit sphere

run time $n^{O(d)}$ *(semidefinite programming)*

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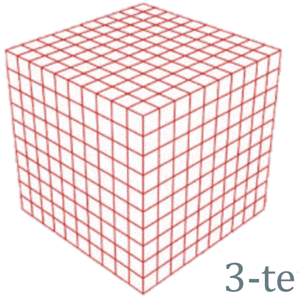
efficient upper bounds on random polynomials



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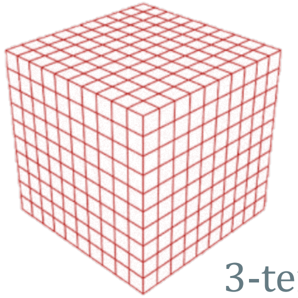

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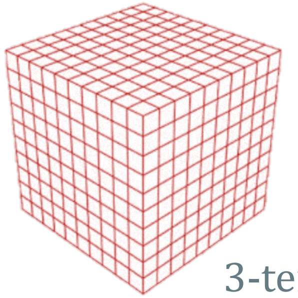

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concretely: $z(x) + \tau_0/2 \cdot (\|x\|^4 - \|x\|^2)$ has matrix representation with $\lambda_{\max}(\cdot) \leq \tau_0$

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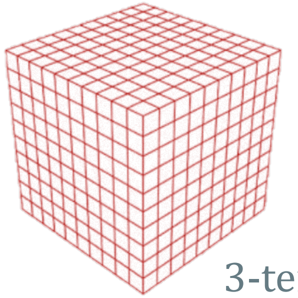

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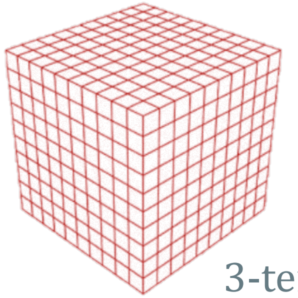

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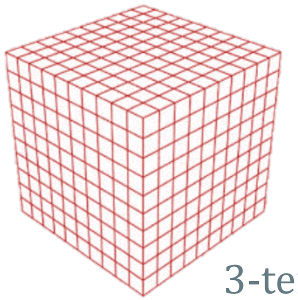

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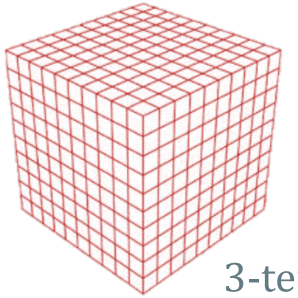

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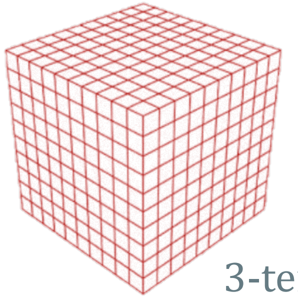

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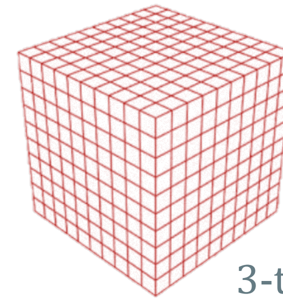

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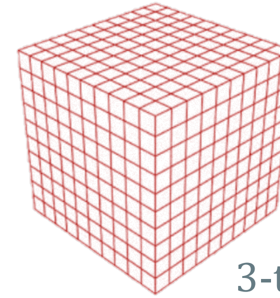

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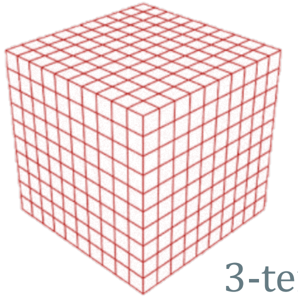
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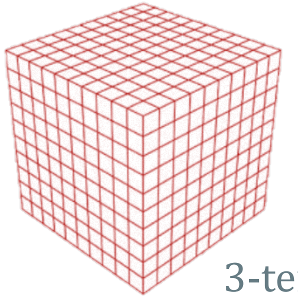

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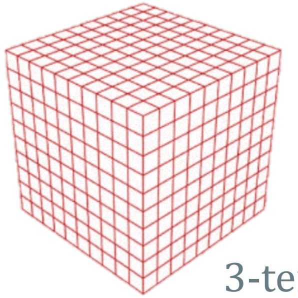

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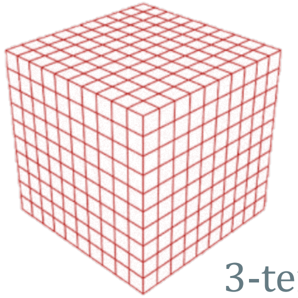

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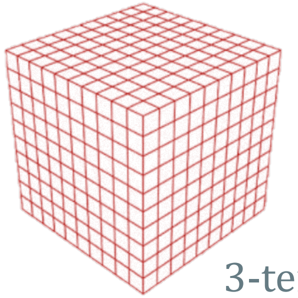

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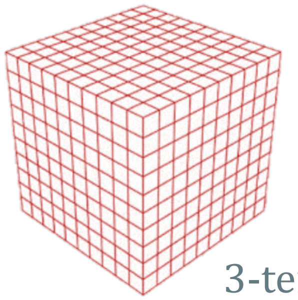

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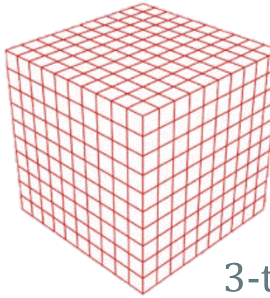

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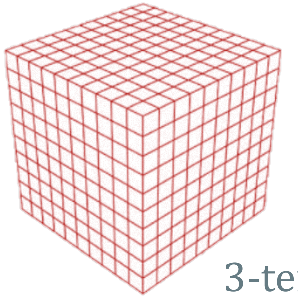

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