Core Decomposition of Uncertain Graphs

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Introduction

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Dense subgraphs

- finding dense subgraphs is a fundamental primitive in many graph problems
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- different definitions of dense subgraphs: cliques, n-cliques, n-clans, k-plexes, k-cores, etc.
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- different definitions of dense subgraphs: cliques, n-cliques, n-clans, k-plexes, k-cores, etc.

- most of them are computationally prohibitive: **NP-hard** or at least quadratic
k-core decomposition

- core decomposition is particularly appealing:
  - it can be computed in linear time
  - it relates to many definitions of dense subgraphs
k-core decomposition

- $G = (V, E)$ is an undirected graph
- **k-core** of $G$ is a maximal subgraph $H = (C, E | C)$ such that $\forall v \in C : \deg_H(v) \geq k$
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![Diagram of k-core decomposition](image)
**k-core decomposition**

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![Graph Diagram](image)
k-core decomposition

- $G = (V, E)$ is an undirected graph
- **k-core** of $G$ is a maximal subgraph $H = (C, E | C)$ such that $\forall v \in C : \text{deg}_H(v) \geq k$

```
   o---o---o
    ^    |
    |    v
   k=2
```
k-core decomposition

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- **k-core** of $G$ is a maximal subgraph $H = (C, E | C)$ such that $\forall v \in C : \deg_H(v) \geq k$

- **core index** of a vertex $v$ is the highest order of a core that contains $v$
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Core Decomposition of Uncertain Graphs
Uncertain graphs

- Many real live networks are associated with uncertainty:
  - data collection process
  - employed machine-learning methods
  - privacy-preserving reasons

- biological networks, protein-interaction networks
- social networks
Uncertain graphs

- Edges in an **uncertain graph** are associated with a probability of existence
Uncertain graphs

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- **Uncertain graph** is a generative model for deterministic graphs
Uncertain graphs

- $G = (V, E, p)$ be an uncertain graph:

$p : E \rightarrow (0, 1]$ is a function that assigns a probability of existence to each edge.
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\[ p : E \to (0, 1] \]

is a function that assigns a probability of existence to each edge.

\[
\Pr(G) = \prod_{e \in E_G} p_e \prod_{e \in E \setminus E_G} (1 - p_e)
\]
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Core Decomposition of Uncertain Graphs

We want to extend the graph tool of core decomposition to the context of uncertain graphs.
Complications

- The fact that core decomposition can be performed in linear time in deterministic graphs does not guarantee efficiency in uncertain graphs.
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- The fact that core decomposition can be performed in linear time in deterministic graphs does not guarantee efficiency in uncertain graphs.

- Are any two vertices connected?
  - in deterministic graph: a simple scan of the graph
  - in uncertain graph: computing the probability that two vertices are connected is a \#P-complete problem
Probabilistic \((k, \eta)\)-cores

- uncertain graph \(G = (V, E, p)\)
- threshold of uncertainty \(\eta \in [0, 1]\)

**Probabilistic \((k, \eta)\)-core** of \(G\) is a maximal subgraph \(H = (C, E \mid C, p)\) such that \(\forall v \in C : \Pr[\deg_H(v) \geq k] \geq \eta\)
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- \[d_v = 3\]

- e.g. \(\Pr[\text{deg}(v) \geq 2] = \Pr[\text{deg}(v) = 2] + [\text{deg}(v) = 3] = (0.1 \times 0.5 \times 0.6 + 0.1 \times 0.4 \times 0.5 + 0.5 \times 0.4 \times 0.9) + (0.5 \times 0.1 \times 0.4)\)
Probabilistic \((k, \eta)\)-cores

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\[
\Pr[\text{deg}(v) \geq k] = \sum_{i=k}^{d_v} \Pr[\text{deg}(v) = i] = 1 - \sum_{i=0}^{k-1} \Pr[\text{deg}(v) = i]
\]

This probability is monotonically non-increasing with \(k\)
Probabilistic \((k, \eta)\)-cores

- \(\eta\)-degree of any vertex \(v \in V\) is defined as
  \[
  \eta\text{-deg}(v) = \max \{ k \in [0..d_v] \mid \Pr[\deg(v) \geq k] \geq \eta \}
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- We use \(\eta\)-degree to define \((k, \eta)\)-core decomposition in a similar manner as degree in deterministic case.
Computing probabilistic cores

- We have proven uniqueness and existence of \((k, \eta)\)-core decomposition of \(G\).
Computing probabilistic cores

- Since naïve computation of $\eta$-degrees leads to exponential time complexity, we defined a dynamic-programming method for $(k, \eta)$-core decomposition.
Computing probabilistic cores

- We have shown the running time of \((k, \eta)\)-core decomposition is \(O(m\Delta)\), where
  - \(m\) is the number of edges
  - \(\Delta\) is the maximum \(\eta\)-degree over all vertices
Computing probabilistic cores

- We have derived a fast-to-compute lower bound on the $\eta$-degree to speed up $(k, \eta)$-core computations.
Applications

1. Task-driven team formation
2. Influence-maximization problem
1. Task-driven team formation problem

- A collaboration graph:
  - vertices are individuals
  - edges exhibit a probabilistic topic model representing the topic(s) of past collaborations
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- A query is a pair \( \langle T, Q \rangle \):
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  - \( Q \) is a set of vertices
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- A query is a pair \( \langle T, Q \rangle \):
  - \( T \) is a set of terms describing a new task
  - \( Q \) is a set of vertices

- The goal is to find an answer set of vertices \( A \), such that \( A \supseteq Q \) is a good team for the task described by \( T \).
2. Influence-maximization problem

- **Independent cascade** (IC) model:
  - Links have associated probability;
  - Every active node $v$ has a single chance of activating each currently inactive neighbor $w$ with probability $p_{vw}$
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- **GREEDY** algorithm adds the vertex bringing the largest marginal gain in the objective function.
2. Influence-maximization problem

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- We reduce the input graph $G$ by some rule and run the **Greedy** algorithm.
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- GREEDY algorithm adds the vertex bringing the largest marginal gain in the objective function.

- We reduce the input graph $G$ by some rule and run the GREEDY algorithm.

- On deterministic graph k-core index is a direct indicator of the expected spread of any vertex (experimentally observed).
Influence-maximization experiment

- Small directed graph from Twitter with influence probabilities learned from past propagations of URLs ($|V| = 21882, |E| = 372005$).
Influence-maximization experiment

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(k, η)-cores-based method outperforms all the baselines.

[Graph showing performance across different criteria and sizes of seed set S]
Conclusions

- We have extended the graph tool of core decomposition to the context of uncertain graphs.
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- We have defined the \((k, \eta)\)-core concept, and devised efficient algorithms for computing a \((k, \eta)\)-core decomposition.
Conclusions

- We have extensively evaluated our definitions and methods on a number of real-world datasets and applications.
Conclusions

- We plan to investigate the relationship between \((k, \eta)\)-cores and other definitions of (probabilistic) dense subgraphs.
Questions?