Prediction by random-walk perturbation

Luc Devroye, Gábor Lugosi and Gergely Neu
Example: sequential routing

For each packet $t = 1, 2, \ldots, T$
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For each packet $t = 1, 2, \ldots, T$
- Choose some path from $u$ to $w$
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- Choose some path from $u$ to $w$
- Packet arrives to $w$ with some delay
Example: sequential routing

For each packet \( t = 1, 2, \ldots, T \):

- Choose some path from \( u \) to \( w \)
- Packet arrives to \( w \) with some delay
- Observe delay on each edge
Switching costs

- Assume that learner has to pay a price of $K > 0$ when switching between routes.
Switching costs

• Assume that learner has to pay a price of $K > 0$ when switching between routes
  – Include header with each packet:

| Header | Data |
Switching costs

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• Assume that learner has to pay a price of $K > 0$ when switching between routes
  – Include header with each packet:
  – Include header only when switching between routes:
Online combinatorial optimization

- **Parameters:** decision set $S = \{v(1), ..., v(N)\} \subseteq \{0,1\}^d$, $m = \max_i \|v(i)\|_1$.

- **For each** $t = 1, 2, ..., n$, **repeat**
  1. Learner chooses action $V_t \in S$.
  2. Environment chooses loss vector $\ell_t \in \mathbb{R}^d$.
  3. Learner suffers loss $V_t^T \ell_t$.
  4. Learner observes $\ell_t$. 
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"Length of the longest path"
Online combinatorial optimization

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**Goal:** minimize regret...

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R_n = \max_{v \in S} \sum_{t=1}^{n} (V_t - v)^T \ell_t
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**Goal:** minimize regret...

\[
R_n = \max_{v \in S} \sum_{t=1}^{n} (V_t - v)^T \ell_t
\]

...while controlling

\[
C_n = \sum_{t=1}^{n} 1_{\{V_t \neq V_{t+1}\}}
\]
## Previous results

### Experts case \((m = 1)\)

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<th>Algorithm</th>
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### Combinatorial case \((m > 1)\)

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Not good if \(d \gg m\)

Not always efficient
Follow the perturbed leader (Kalai & Vempala, 2005)

• In each round $t = 1, 2, ..., n$
  1. Set $L_{t-1} = \sum_{s=1}^{t-1} \ell_s$.
  2. Draw some i.i.d. perturbations $Z_t \in \mathbb{R}^d$.
  3. Choose action
     $$V_t = \arg\min_{v \in S} v^T (L_{t-1} + Z_t).$$
Prediction by random-walk perturbation

- Let $Z_{0,i} = 0$ for all $i = 1,2, \ldots, d$
- In each round $t = 1,2, \ldots, n$
  1. Set $L_{t-1} = \sum_{s=1}^{t-1} \ell_s$.
  2. Draw $X_{t,i} = \pm 1/2$ with equal probabilities
  3. Let

\[
Z_t = Z_{t-1} + X_t = \sum_{s=1}^{t} X_s
\]

  4. Choose action

\[
V_t = \arg\min_{v \in S} v^\top (L_{t-1} + Z_t)
\]
Prediction by random-walk perturbation

• Let $Z_{0,i} = 0$ for all $i = 1, 2, \ldots, d$

• In each round $t = 1, 2, \ldots, n$
  1. Set $L_{t-1} = \sum_{s=1}^{t-1} \ell_s$.
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    $$Z_t = Z_{t-1} + X_t = \sum_{s=1}^{t} X_s$$

  4. Choose action

    $$V_t = \arg\min_{v \in S} v^\top (L_{t-1} + Z_t)$$

Replace i.i.d. perturbations by symmetric random walks
Analysis for $m = 1$

Lemma: $E[R_n] \leq 2E[C_n]$
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- Intuition: setting $\ell_{t,i} = 0$ gives the most switches
Analysis for $m = 1$

**Lemma:** $\mathbb{E}[R_n] \leq 2\mathbb{E}[C_n]$

- **Intuition:** setting $\ell_{t,i} = 0$ gives the most switches
- **Hope:** For $d = 2$, $C_n = O_p(\sqrt{n})$
Number of switches, $d = 10$
Number of switches, $d = 50$
Number of switches, $d = 100$
Number of switches, $d = 100$

Lead pack: $A_t = \{i : Z_{i,t} > \max_j Z_{j,t} - 2\}$
Number of switches, $d = 100$

**Lead pack:** $A_t = \{i: Z_{i,t} > \max_j Z_{j,t} - 2\}$

**Lemma:**

$$P[|A_t| > 1] \leq \frac{8}{t} + 4 \sqrt{\frac{2 \log d}{t}}$$
Proof idea \((m = 1)\)

- For simplicity, let \(X'_{t,i} \sim N(0,1)\), so \(Z'_{t,i} \sim N(0, t)\)
- Let \(c = \|X'_{t+1}\|_{\infty}\)
Proof idea ($m = 1$)

• For simplicity, let $X'_{t,i} \sim N(0,1)$, so $Z'_{t,i} \sim N(0, t)$
• Let $c = \|X'_{t+1}\|_\infty$
• PDF of $\max_i Z'_{t,i}: \varphi_t(x)$

\[
P_t[|A'_t| > 1] \leq \int_{-\infty}^{\infty} \varphi_t^*(x) \left( \frac{c^2}{2t} + \frac{cx}{t} \right) dx
\]

\[
= \frac{c^2}{2t} + \frac{c}{t} \mathbb{E} \left[ \max_i Z_{t,i} \right] \leq \frac{c^2}{2t} + c \sqrt{2 \log d \over t}
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$$

Can be tightened when using Rademacher perturbations
Main result for $m = 1$

Theorem:

$$\mathbb{E}[R_n] \leq 2\mathbb{E}[C_n] \leq 8\sqrt{2n \log d} + 16 \log n + 16$$
Combinatorial action set ($m > 1$)

- Use $X_{i,t} \sim N(0,1)$ for all $i, t$
Combinatorial action set \((m > 1)\)

- Use \(X_{i,t} \sim N(0,1)\) for all \(i, t\)
- Lead pack: \(A_t \subseteq S\) such that for any \(v \not\in A_t,\)
  \[P_t[V_{t+1} = v] = 0\]
Combinatorial action set \((m > 1)\)

- Use \(X_{i,t} \sim N(0,1)\) for all \(i, t\)
- Lead pack: \(A_t \subseteq S\) such that for any \(v \notin A_t\),
  \[ P_t[V_{t+1} = v] = 0 \]

\[ P[|A_t| > 1] \leq \frac{m \log d}{\sqrt{t}} + \frac{m \log d}{2t} \]
Proof idea ($m > 1$)

- Let $c = \|X_{t+1}\|_\infty$
- PDF of $\max_{\nu \in \mathcal{S}} \nu^\top Z_t: \phi^*_t(x)$

\[
P_t[|A_t| > 1] \leq \int_{-\infty}^{\infty} \phi^*_t(x) \left( \frac{mc^2}{2t} + \frac{cx}{t} \right) dx
\]

\[
= \frac{mc^2}{2t} + \frac{c}{t} \mathbb{E} \left[ \max_{\nu \in \mathcal{S}} \nu^\top Z_t \right] \leq \frac{mc^2}{2t} + mc \sqrt{\frac{2 \log d}{t}}
\]
Main result for $m > 1$

**Theorem:**

$$E[C_n] = O(m \log d \sqrt{n})$$
$$E[R_n] = O(m^2 \log d \sqrt{n})$$

Both only logarithmic in $d$
Conclusions & open problems

• Algorithm has absolutely no parameters!
  – Tune it to get $\mathbb{E}[R_n] = O(\sqrt{L_n^*})$?
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• High probability bounds on both $C_n$ and $R_n$?
  – Done for $d = 2$
  – First algorithm with this property
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• Bounding $\mathbb{E}[\|V_{t+1} - V_t\|_1]$ instead of $\mathbb{P}[V_{t+1} \neq V_t]$?
  – Conjecture: $\mathbb{E}[\|V_{t+1} - V_t\|_1] = O(m^{3/2} \log d / \sqrt{n})$ instead of $O(m^2 \log d / \sqrt{n})$
Conclusions & open problems

• Algorithm has absolutely no parameters!
  – Tune it to get \( \mathbb{E}[R_n] = O(\sqrt{L_n^*}) \)?

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• Extension to bandit feedback?