Open problems in throughput scheduling

Jiří Sgall

Computer Science Inst. of the Charles Univ. Prague

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A puzzle

Move boxes within their ranges.
A puzzle

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A puzzle

- Move boxes within their ranges.
- Align them so that they do not overlap vertically.
A puzzle

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- Is this easy (in P) or difficult (NP-hard)?
A puzzle

- Move boxes within their ranges.
- Align them so that they do not overlap vertically.
- Is this easy (in P) or difficult (NP-hard)?
- What if there are only two (or 1000) different sizes of boxes?
Throughput scheduling

- Environment: One or more machines.
- Input: Jobs with length $p_j$, release time $r_j$, deadline $d_j$, and weight $w_j$. (Parameters are integers.)
- Output: Each job is assigned to a machine for a subinterval of $[r_j, d_j)$ of length $p_j$ or rejected. No overlaps.
- Objective: Maximize the number (weight) of the completed jobs.
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This talk

- Online algorithms.
- Usually a single machine.
Throughput scheduling

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This talk

- Online algorithms.
- Usually a single machine.
- Either jobs of equal length ($p_j = p$) and no weights
- or jobs of unit length ($p_j = 1$) with weights.
Online scheduling

- At time $r_j$, the other parameters of the job become known.
- At each time, if a machine is idle, the algorithm may decide to start a job.
Online scheduling

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Competitive ratio

An algorithm $A$ is $R$-competitive if for every instance $I$

- $OPT(I) \leq R \cdot A(I)$ for a deterministic algorithm
Online scheduling

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- At each time, if a machine is idle, the algorithm may decide to start a job.

**Competitive ratio**

An algorithm $A$ is $R$-competitive if for every instance $I$

- $OPT(I) \leq R \cdot A(I)$ for a deterministic algorithm, or
- $OPT(I) \leq R \cdot E[A(I)]$ for a randomized algorithm.
### Other scheduling problems

#### Variants

- **Machine environments**: speeds, shop scheduling (more operations) etc.
- **Job parameters and restrictions**: preemption, dependencies, resources etc.

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**Typical objectives**

- **MinMax**: Minimize the length of schedule (or another global measure of balance).
- **MinSum**: Minimize the average completion time of a job (or waiting time, flow time, stretch, possibly weighted).
Other scheduling problems

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- Machine environments: speeds, shop scheduling (more operations) etc.
- Job parameters and restrictions: preemption, dependencies, resources etc.

Typical objectives

- MinMax: Minimize the length of schedule (or another global measure of balance).
- MinSum: Minimize the average completion time of a job (or waiting time, flow time, stretch, possibly weighted).
Jobs of equal length

Setting

- Equal lengths of jobs ($p_j = p$).
- No weights.
- Single machine.
Jobs of equal length

Setting

- Equal lengths of jobs ($p_j = p$).
- No weights.
- Single machine.

Outline

1. Offline problem is polynomial.
2. Greedy algorithms are $2$-competitive.
3. Lower bounds.
4. A better randomized algorithm.
5. Generalizations, variants.
Greedy algorithms

GREEDY: If idle, start an arbitrary job.
Greedy algorithms

GREEDY: If idle, start an arbitrary job.

Any such algorithm is 2-competitive.
Greedy algorithms

GREEDY: If idle, start an arbitrary job.

Charging scheme – GREEDY is 2-competitive

Charge (map) a job in OPT to itself in GREEDY, if scheduled.
Greedy algorithms

**GREEDY**: If idle, start an arbitrary job.

**Charging scheme** – GREEDY is 2-competitive

- Charge (map) a job in OPT to itself in GREEDY, if scheduled.
- Otherwise charge a job that OPT starts at \( t \) to the job GREEDY runs at \( t \).
Lower bounds

No deterministic algorithm is better than 2-competitive. No randomized algorithm is better than $\frac{4}{3}$-competitive. (For one of the two instances, on average, runs at most 1.5 jobs out of 2.)
No deterministic algorithm is better than 2-competitive.

No randomized algorithm is better than $\frac{4}{3}$-competitive. (For one of the two instances, on average, runs at most 1.5 jobs out of 2.)
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No randomized algorithm is better than $4/3$-competitive. (For one of the two instances, on average, runs at most 1.5 jobs out of 2.)
A barely random algorithm I

- Generate two schedules, A and B. Flip a coin to choose one of them.
A barely random algorithm I

- Generate two schedules, A and B. Flip a coin to choose one of them.
- A and B are produced by two identical processes using a common lock.
A barely random algorithm I

- Generate two schedules, A and B. Flip a coin to choose one of them.

- A and B are produced by two identical processes using a common lock.

- If the machine is idle (in A or B) and the set of pending jobs is not flexible (idling for time $p$ would lose some job), start the most urgent job.

- If the machine is idle (in A or B) and the set of pending jobs is flexible (idling for time $p$ does no harm):
  - If the lock is available, acquire it, start the most urgent job and release the lock after the job is completed.
  - Otherwise stay idle.
A barely random algorithm II

LOCK, A

B
A barely random algorithm II

LOCK, A
B

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A barely random algorithm II

A

B
A barely random algorithm II

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A barely random algorithm II

Equal length jobs  Unit time jobs  Offline

Greedy  Lower bounds  Randomized  Variants

A barely random algorithm II

LOCK, B

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A barely random algorithm II
A barely random algorithm II

A barely random algorithm II
A barely random algorithm II
A barely random algorithm III

- Analyzed by a more complex charging scheme.
- Each job in OPT charges $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{6}$ to itself or to the job running at the same time in A and B.
- Each job in A or B is charged at most $\frac{5}{6}$.

Theorem

This algorithm is $\frac{5}{3}$-competitive.

Open problem

Find a randomized algorithm with the optimal competitive ratio.
A barely random algorithm III

- Analyzed by a more complex charging scheme.
- Each job in OPT charges $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{6}$ to itself or to the job running at the same time in A and B.
- Each job in A or B is charged at most $\frac{5}{6}$.

**Theorem**

*This algorithm is $\frac{5}{3}$-competitive.*
A barely random algorithm III

- Analyzed by a more complex charging scheme.
- Each job in OPT charges $1/2$, $1/3$, or $1/6$ to itself or to the job running at the same time in A and B.
- Each job in A or B is charged at most $5/6$.

**Theorem**

This algorithm is $5/3$-competitive.

**Open problem**

Find a randomized algorithm with the optimal competitive ratio.
Parallel machines make the problem easier!
Parallel machines make the problem easier!

**Results**

- For 2 machines, there is a $3/2$-competitive deterministic algorithm and this is optimal.
- For $m$ machines, there is an $R$-competitive deterministic algorithm with $R \rightarrow e/(e - 1) \approx 1.58$ for $m \rightarrow \infty$.
- The lower bound approaches $6/5$ for $m \rightarrow \infty$.
More machines

Parallel machines make the problem easier!

Results

- For 2 machines, there is a $3/2$-competitive deterministic algorithm and this is optimal.
- For $m$ machines, there is an $R$-competitive deterministic algorithm with $R \to e/(e-1) \approx 1.58$ for $m \to \infty$.
- The lower bound approaches $6/5$ for $m \to \infty$.

Open problem

Decrease the gap for $m \to \infty$. 
Jobs with fixed start times

- Each job has to be started at its release $r_j$ or rejected.
- Jobs have a length $p_j$ and a weight $w_j$.
- Jobs can be stopped (preempted).
Jobs with fixed start times

- Each job has to be started at its release $r_j$ or rejected.
- Jobs have a length $p_j$ and a weight $w_j$.
- Jobs can be stopped (preempted).

Results

- There is a 4-competitive algorithm for various cases, including equal times ($p_j = p$), unit weights ($w_j = 1$), and uniform weights ($w_j = p_j$); it works for parallel machines.
- There is a matching lower bound.
Machines with speeds

- Each job has to be started at its release $r_j$ or rejected.
- A machine with speed $s_i$ processes job $j$ in time $p_j/s_i$. 

Open problems in throughput scheduling
Machines with speeds

- Each job has to be started at its release $r_j$ or rejected.
- A machine with speed $s_i$ processes job $j$ in time $p_j/s_i$.
- Jobs are identical ($p_j = 1$ and $w_j = 1$).

GREEDY: Start the released job on the fastest available machine.
Machines with speeds

- Each job has to be started at its release $r_j$ or rejected.
- A machine with speed $s_i$ processes job $j$ in time $p_j/s_i$.
- Jobs are identical ($p_j = 1$ and $w_j = 1$).

GREEDY: Start the released job on the fastest available machine.

Results for the greedy algorithm

- For two machines, GREEDY is $4/3$-competitive and this is optimal.
- For $m \to \infty$ the competitive ratio is between 1.56 and 2.

Open problem(s)

Analyze GREEDY, or find another algorithm with a competitive ratio below 2.
Unit time jobs with weights

Setting

- Unit length of jobs ($p_j = 1$).
- General weights.
- Single machine.
Unit time jobs with weights

Setting

- Unit length of jobs ($p_j = 1$).
- General weights.
- Single machine.

Outline

1. Offline problem is easy (matching).
2. Greedy algorithm is 2-competitive.
3. A better randomized algorithm.
4. A better deterministic algorithm.
5. Generalizations, variants.
Motivation and variants

Forwarding packets in network switches
Motivation and variants

Forwarding packets in network switches

Restricted scenarios

- 2-bounded: Some packets may wait a single step, some packets not at all. \((d_j \leq r_j + 2)\)
- Agreeable deadlines: \(r_j < r_k\) implies \(d_j \leq d_k\).
- Weighted queues: The deadlines are not known, only their order.
- Limited number of weights.
**Greedy algorithm**

**GREEDY:** If idle, start a pending job with the maximal weight.

- **21**
- **20**
- **11**
- **10**
- **5**

Charging scheme – **GREEDY** is 2-competitive.
Charge (map) a job in **OPT** to itself in **GREEDY**, if scheduled. Otherwise charge a job in **OPT** to the job **GREEDY** runs at the same time.
**GREEDY** algorithm

If idle, start a pending job with the maximal weight.

**GREEDY**

- Jobs: 21, 11, 5

**OPT**

- Jobs: 20, 10, 11, 5, 21
Greedy algorithm

**GREEDY**: If idle, start a pending job with the maximal weight.

**Charging scheme** – GREEDY is 2-competitive

- Charge (map) a job in OPT to itself in GREEDY, if scheduled.
- Otherwise charge a job in OPT to the job GREEDY runs at the same time.
A randomized algorithm

- At each time, pick uniformly random real $x \in (-1, 0)$.
- Let $h$ be the largest weight of a pending job.
- Among all the pending jobs with $w_j \geq e^x \cdot h$, schedule a job with the earliest deadline.

Theorem

This algorithm is $e/(e-1) \approx 1.58$-competitive.
How much “money” we need at a given time and configuration?

We earn \( R \cdot w_j \) for running a job and pay \( w_j \) if OPT runs a job.
A potential function

How much “money” we need at a given time and configuration?

We earn $R \cdot w_j$ for running a job and pay $w_j$ if OPT runs a job.

Let $\Phi = \sum_{j \in X} w_j$, where $X$ are the jobs that the algorithm completed but the adversary will schedule in the future.
A potential function

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To prove that ON is $R$-competitive, we show that in each step

$$\Phi_{old} + R \cdot w_{ON} - w_{OPT} \geq \Phi_{new}$$
A potential function

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Let $\Phi = \sum_{j \in X} w_j$, where $X$ are the jobs that the algorithm completed but the adversary will schedule in the future.

To prove that ON is $R$-competitive, we show that in each step

$$\Phi_{old} + R \cdot E[w_{ON}] - w_{OPT} \geq E[\Phi_{new}]$$
At each time, pick uniformly random real $x \in (-1, 0)$.
Let $h$ be the largest weight of a pending job.
Among all the pending jobs with $w_j \geq e^x \cdot h$, schedule a job with the earliest deadline.

\[ \Phi_{\text{old}} + R \cdot E[w_{\text{ON}}] - w_{\text{OPT}} \geq E[\Phi_{\text{new}}] \]

**Theorem**

*This algorithm is $e/(e - 1) \approx 1.58$-competitive. This is optimal against the adaptive online adversary. I.e., it is optimal among the algorithms analyzed using a potential.*
Deterministic algorithms I

Charging scheme

Alternating heavy and urgent packets eventually leads to a $1.939$-competitive algorithm.
Deterministic algorithms I

Charging scheme

Alternating heavy and urgent packets eventually leads to a 1.939-competitive algorithm.

Potential function

Can be used to give a 1.828-competitive algorithm.
Modifying the optimal schedule

At each step, the configuration of the optimal schedule is made identical with that of the online algorithm, with some advantage to the optimum:

- Schedule a job and keep it pending,
- Schedule two jobs,
- Increase the weight or deadline of some pending job.

Can be used to give a $\phi \approx 1.618$-competitive algorithm for instances with agreeable deadlines.
Deterministic algorithms II

Modifying the optimal schedule

At each step, the configuration of the optimal schedule is made identical with that of the online algorithm, with some advantage to the optimum:

- Schedule a job and keep it pending,
- Schedule two jobs,
- Increase the weight or deadline of some pending job.

Can be used to give a $\phi \approx 1.618$-competitive algorithm for instances with agreeable deadlines.

Weighted queues

There exists a 1.897-competitive algorithm.
Lower bounds

2-bounded instances

- The $\phi \approx 1.618$-competitive deterministic algorithm is optimal.
- There exists a $1.25$-competitive randomized algorithm and this is optimal.

No other lower bounds for the general problem are known.
Lower bounds

2-bounded instances

- The $\phi \approx 1.618$-competitive deterministic algorithm is optimal.
- There exists a $1.25$-competitive randomized algorithm and this is optimal.

No other lower bounds for the general problem are known.

Open problem

Is the general problem harder than the 2-bounded case?
Offline scheduling

- For unrestricted job lengths, the problem is strongly NP-hard.
- For unit jobs \((p_j = 1)\) and arbitrary weights we can maximize the weight of scheduled jobs in polynomial time.
Even maximizing the weight for equal-length jobs \((p_j = p)\) on a single machine is in \(P\).

For unit jobs and one more job length \((p_j = \{1, p\})\), we can test if all the jobs can be scheduled.
A linear program

Variables: $x_t$ – the number of long jobs started before time $t$.
Constraints: For all times $s, t$:

\[
\begin{align*}
    x_t - x_{t-1} & \geq 0 \\
    x_t - x_{t-p} & \leq 1 \\
    x_{t+1-p} - x_s & \geq b_{s,t} \\
    x_{t+1-p} - x_s & \leq \lfloor (t - s - a_{s,t})/p \rfloor
\end{align*}
\]

where $a_{s,t}$ and $b_{s,t}$ is the number of short and long jobs, resp., that have to start and complete in $[s, t)$.
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Observation

The matrix of the LP is totally unimodular. Thus if the LP is feasible, then there exists an integral solution.
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Variables: $x_t$ – the number of long jobs started before time $t$.
Constraints: For all times $s, t$:

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$$x_{t+1-p} - x_s \leq \lfloor (t - s - a_{s,t})/p \rfloor$$

where $a_{s,t}$ and $b_{s,t}$ is the number of short and long jobs, resp., that have to start and complete in $[s, t)$.

- A schedule implies a feasible (integral) solution: Easy.
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where $a_{s,t}$ and $b_{s,t}$ is the number of short and long jobs, resp., that have to start and complete in $[s, t)$.

- A schedule implies a feasible (integral) solution: Easy.
- A feasible integral solution implies a schedule: Subtle, holds only for a single machine.
A linear program

Variables: $x_t$ – the number of long jobs started before time $t$.
Constraints: For all times $s, t$:

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\begin{align*}
    x_t - x_{t-1} & \geq 0 \\
    x_t - x_{t-p} & \leq 1 \\
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    x_{t+1-p} - x_s & \leq \lfloor (t - s - a_{s,t}) / p \rfloor
\end{align*}
\]

where $a_{s,t}$ and $b_{s,t}$ is the number of short and long jobs, resp., that have to start and complete in $[s, t)$.
Offline scheduling – open problems

Open problems

- If \( p_j \in \{2, 3\} \), is it polynomial to decide if all jobs can be scheduled?
- If \( p_j \in \{1, 2\} \), is it polynomial to maximize the number of scheduled jobs?
Offline scheduling – open problems

Open problems

- If $p_j \in \{2, 3\}$, is it polynomial to decide if all jobs can be scheduled?
- If $p_j \in \{1, 2\}$, is it polynomial to maximize the number of scheduled jobs?
- For some constant $C$, is it NP-hard to maximize the weight of the scheduled jobs on instances with $p_j \leq C$ for all jobs?
Offline scheduling – open problems

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THANK YOU!