Alan Turing
and
Number Theory

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http://logic.pdmi.ras.ru/~yumat
Recognition of Merits
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Andrew R. Booker.
Artin’s Conjecture, Turing’s Method, and the Riemann Hypothesis. 
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Artin’s Conjecture, Turing’s Method, and the Riemann Hypothesis.

“Reading Turing’s paper on the subject, which was one of his last, one marvels at what he accomplished with the limited computational resources of the day. His method was truly ahead of its time.”
Distribution of Prime Numbers

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\[ \pi(x) - \frac{x}{\ln(x)} , \quad \pi(x) - \text{Li}(x) \]

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### Numerical Values

<table>
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<tr>
<th>$x$</th>
<th>$\pi(x) - x/\log(x)$</th>
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**Hardy:** "the largest number that have served any definite purpose in mathematics"
Ingham’s Book

A. E. Ingham.
The Distribution of Prime Numbers.
Cambridge University Press, 1932

Reissued in 1990
Riemann’s zeta function

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Dirichlet series:

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]

\[ s = \sigma + it \]

The series converges in the semi-plane \( \Re(s) > 1 \) and defines a function that can be analytically extended to the entire complex plane except for the point \( s = 1 \), its only (and simple) pole.
The Riemann Hypothesis

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**Equivalent formulation of RH.**
\[ \pi(x) - \text{Li}(x) = O(\sqrt[4]{x} \log(x)) \].
The Riemann Hypothesis

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**Equivalent formulation of RH.**
\[ \pi(x) - \text{Li}(x) = O(x^{\frac{1}{2}} \log(x)). \]

**Theorem (Miller [1975]).** *If the Generalized Riemann Hypothesis is true, then* \( \text{Primes} \subset \mathbb{P}. \)
## Numerical checking of RH

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<td>J. I. Hutchinson</td>
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<tr>
<td>1936</td>
<td>1041</td>
<td>E. C. Titchmarsh</td>
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\pi(x) > \text{Li}(x) \quad (\ast)
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x < 10^{10^{10^{34}}}.
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**Theorem (Skewes [1955]).** If the Riemann Hypothesis is false, then there exists an \( x \) satisfying (\( \ast \)) such that

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x < 10^{10^{10^{10^3}}}.
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1. It is proposed to make calculations of the Riemann zeta-function on the critical line for $1,450 < t < 6,000$ with a view to discovering whether all the zeros of the function in this range of $t$ lie on the critical line. An investigation for $0 < t < 1,464$ has already been made by Titchmarsh. The most laborious part of such calculations consists in the evaluation of certain trigonometrical sums

$$\sum_{r=1}^{m} \frac{1}{\sqrt{r}} \cos (t \log r - \theta)$$

$$m = \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor$$
In the present calculation it is intended to evaluate these sums approximately in most cases by the use of apparatus somewhat similar to what is used for tide prediction. When this method does not give sufficient accuracy it will be necessary to revert to the straightforward calculation of the trigonometric sums, but this should be only very rarely necessary. I am hoping that the use of the tide-predicting machine will reduce the amount of such calculation necessary in a ratio of 50:1 or better. It will not be feasible to use already existing tide predictors because the frequencies occurring in the tide problem are entirely different from those occurring in the zeta-function problem. I shall be

\[ \sum_{r=1}^{m} \frac{1}{\sqrt{r}} \cos (t \log r) \]
The cost from those occurring in the zeta-function problem. I shall be working in collaboration with D. C. MacPhail, a research student who is an engineer. We propose to do most of the machine-shop work ourselves, and are therefore applying only for the cost of materials, and some preliminary computation.

2. Cost of materials for making tide predictor, estimated at £25, and not exceeding £35. Cost of preliminary computation, estimated at £3 10s., and not exceeding £5. Some further computation may be necessary after the work with the tide predictor, but the amount of this cannot be accurately estimated at this stage, and might be negligible. No application is being made on this account at present.
Tide-predicting Machine

http://tidesandcurrents.noaa.gov/predma3.html
Tide-predicting Machines

King's College,
Cambridge.
24 March 1939.

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March, 1939

A. M. Turing.


The paper was submitted on March 7, 1939

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Methods for the calculation of the zeta-function

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Euler–Maclaurin summation

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"Only one other method, besides the Euler–Maclaurin and Riemann–Siegel ones, seems to have been proposed for computing $\zeta(s)$ to moderate accuracy at large heights, namely the one due to Turing. It was designed to provide higher accuracy than was guaranteed by the crude bounds on the remainder term in the Riemann–Siegel formula that were available at that time, and at the same time be more efficient than the Euler–Maclaurin formula. However, very good estimates for remainder terms in the Riemann–Siegel formula are now available, which seem to make Turing's method unnecessary."
Introduction
In June 1950 the Manchester University Mark 1 Electronic Computer was used to do some calculations concerned with the distribution of the zeros of the Riemann zeta-function. It was intended in fact to determine whether there are any zeros not on the critical line in certain particular intervals.
SOME CALCULATIONS OF THE RIEMANN ZETA-FUNCTION

By A. M. TURING

[Received 29 February 1952.—Read 20 March 1952]

Introduction

In June 1950 the Manchester University Mark 1 Electronic Computer was used to do some calculations concerned with the distribution of the zeros of the Riemann zeta-function. It was intended in fact to determine whether there are any zeros not on the critical line in certain particular intervals.
The calculations were done in an optimistic hope that a zero would be found off the critical line, and the calculations were directed more towards finding such zeros than proving that none existed. The procedure was such that if it had been accurately followed, and if the machine made no errors in the period, then one could be sure that there were no zeros off the critical line in the interval in question. In practice only a few of the results were checked by repeating the calculation, so that the machine might well have made an error.
If more time had been available it was intended to do some more calculations in an altogether different spirit. There is no reason in principle why computation should not be carried through with the rigour usual in mathematical analysis. If definite rules are laid down as to how the computation is to be done one can predict bounds for the errors throughout. When the computations are done by hand there are serious practical difficulties about this. The computer will probably have his own ideas as to how certain steps should be done. When certain steps may be omitted without serious loss of accuracy he will wish to do so. Furthermore he will probably not see the point of the ‘rigorous’ computation and will probably say ‘If you want more certainty about the accuracy why not just take more figures?’ an argument difficult to counter. However, if the calculations are being done by an automatic computer one can feel sure that this kind of indiscipline does not occur. Even with the automatic computer this rigour can be rather tiresome to achieve, but in connexion with such a subject as the analytical theory of numbers, where rigour is the essence, it seems worth while.
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Cauchy integral

Let $N(T)$ denote the number of zeros of $\zeta(s)$ inside the rectangular region

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$$N(T) = \frac{1}{2\pi i} \oint \frac{\zeta'(s)}{\zeta(s)} \, ds$$
Alongside the Critical Line

\[ Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right) \]
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Classical Method for Checking Riemann’s Hypothesis

Calculate $N(T) = \frac{1}{2} \pi i \oint \zeta'(s) \zeta(s) ds$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1$, $1 \leq \Im(s) \leq T$.

Select $N(T) + 1$ numbers (“green points”) $f_0, \ldots, f_{N(T)}$ such that $1 \leq f_0 < \cdots < f_k < f_{k+1} < \cdots < f_{N(T)} \leq T$ and verify that $Z(f_{k-1} - 1) Z(f_k) < 0$, $k = 1, \ldots, N(T)$.

In case of success all zeros of $\zeta(s)$ with imaginary part between 1 and $T$ do lie exactly on the critical line $\Re(s) = \frac{1}{2}$.
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and verify that

\[
Z(f_{k-1})Z(f_k) < 0, \quad k = 1, \ldots, N(T).
\]

In case of success all zeros of \( \zeta(s) \) with imaginary part between 1 and \( T \) do lie **exactly** on the critical line \( \Re(s) = \frac{1}{2} \).
Gram Points

\[ Z(t) = e^{i\theta(t)} \zeta \left( \frac{1}{2} + it \right) \]

\[ \theta(t) = \Im \ln \left( \Gamma\left( \frac{it}{2} + \frac{1}{4} \right) \right) - \frac{t}{2} \ln(\pi) \]
Gram Points

\[ Z(t) = e^{i\theta(t)} \zeta \left( \frac{1}{2} + it \right) \]

\[ \theta(t) = \Im \ln \left( \Gamma \left( \frac{it}{2} + \frac{1}{4} \right) \right) - \frac{t}{2} \ln(\pi) \]

\[ \theta(t) = \frac{1}{2} \left( \ln \left( \frac{t}{2\pi} \right) - 1 \right) t - \frac{\pi}{8} + o(t^{-1}) \]
Gram Points

\[ Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right) \]

\[ \theta(t) = \Im \ln \left( \Gamma\left(\frac{it}{2} + \frac{1}{4}\right) \right) - \frac{t}{2} \ln(\pi) \]

\[ e^{i\theta(t)} = \cos(\theta(t)) + i \sin(\theta(t)) \]
Gram Points

\[ Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right) \]
\[ \theta(t) = \Im \ln \left( \Gamma\left(\frac{it}{2} + \frac{1}{4}\right) \right) - \frac{t}{2} \ln(\pi) \]
\[ \sin(\theta(t)) \]

![Graph showing the relationship between the Gram points and the function described in the text.](image-url)
Gram Points

\[ Z(t) = e^{i \theta(t)} \zeta \left( \frac{1}{2} + it \right) \]
\[ \sin(\theta(t)) \]
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\[ \sin(\theta(t)) \quad \sin(\theta(g_m)) = 0 \]
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\[ \sin(\theta(t)) \quad \sin(\theta(g_m)) = 0 \quad \theta(g_m) = \pi m \]
Gram’s “law”

\[ Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right) \]

\[ \sin(\theta(t)) = \sin \left( \Im \ln \left( \Gamma\left(\frac{it}{2} + \frac{1}{4}\right) \right) - \frac{t}{2} \ln(\pi) \right) \]
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Gram’s “Law”. \((-1)^m Z(g_m) > 0\)
Violation of Gram’s “Law”

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\[ Z(g_{133}) = -0.7763 \ldots \]

\[ Z(g_{135}) = -3.4698 \ldots \]
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Bad Gram point: \((-1)^m Z(g_m) < 0\)
Violation of Gram’s “Law”

\[ Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2} + it) \]

\[ \sin(\theta(t)) = \sin \left( \frac{\varphi}{\ln \left( \Gamma \left( \frac{it}{2} + \frac{1}{4} \right) \right) - \frac{t}{2} \ln(\pi)} \right) \]

Gram’s “Law”. \((-1)^m Z(g_m) > 0\)

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\[ Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right) \quad \sin(\theta(t)) = \sin \left( \mathcal{G} \ln \left( \Gamma \left( \frac{it}{2} + \frac{1}{4} \right) \right) - \frac{t}{2} \ln(\pi) \right) \]

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Gram’s “Law”. \((-1)^m Z(g_m) > 0\)

Z(g_{133}) = -0.7763 \ldots \quad Z(g_{134}) = -0.0169 \ldots \quad Z(g_{135}) = -3.4698 \ldots \\
Z(g_{134} + h_{134}) = +0.1958 \ldots
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\[ \sin(\theta(t)) = \sin\left(\Im \ln \left( \Gamma\left(\frac{it}{2} + \frac{1}{4}\right) \right) - \frac{t}{2} \ln(\pi) \right) \]

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\[ Z(g_{134} + h_{134}) = +0.1958 \ldots \]

\[ g_{133} < g_{134} + h_{134} < g_{135} \quad h_{134} = -0.2 \]
Classical Method for Checking Riemann’s Hypothesis

Calculate $N(T) = \frac{1}{2} \pi \iota \oint \zeta'(s) \zeta(s) \, ds$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1$, $1 \leq \Im(s) \leq T$.

Find $N(T) + 1$ "small" numbers $h^{-1}, \ldots, h_{N(T)-1}$ such that

$\lfloor g^{-1} + h^{-1} < \cdots < g_{k} + h_{k} < \cdots < g_{N(T)-1} + h_{N(T)-1} \rfloor \leq T$

and verify that $Z\left(g_{k}^{-1} + h_{k}^{-1}\right) Z\left(g_{k} + h_{k}\right) < 0$, $k = 0, \ldots, N(T)-1$.

In case of success all zeros of $\zeta(s)$ with imaginary part between 1 and $T$ do lie exactly on the critical line $\Re(s) = \frac{1}{2}$. 
Classical Method for Checking Riemann’s Hypothesis

Calculate $N(T) = \frac{1}{2\pi i} \oint \frac{\zeta'(s)}{\zeta(s)} \, ds$, the number of zeros of $\zeta(s)$ inside the rectangular region

$$0 \leq \Re(s) \leq 1, \quad 1 \leq \Im(s) \leq T.$$
Classical Method for Checking Riemann’s Hypothesis

- Calculate $N(T) = \frac{1}{2\pi i} \oint \frac{\zeta'(s)}{\zeta(s)} \, ds$, the number of zeros of $\zeta(s)$ inside the rectangular region

\[ 0 \leq \Re(s) \leq 1, \quad 1 \leq \Im(s) \leq T. \]

- Find $N(T) + 1$ “small” numbers $h_{-1}, \ldots, h_{N(T)-1}$ such that

\[ 1 \leq g_{-1} + h_{-1} < \cdots < g_k + h_k < g_{k+1} + h_{k+1} < \cdots < g_{N(T)-1} + h_{N(T)-1} \leq T \]

and verify that

\[ Z(g_{k-1} + h_{k-1})Z(g_k + h_k) < 0, \quad k = 0, \ldots, N(T) - 1. \]
Classical Method for Checking Riemann’s Hypothesis

- Calculate $N(T) = \frac{1}{2\pi i} \oint \frac{\zeta'(s)}{\zeta(s)} \, ds$, the number of zeros of $\zeta(s)$ inside the rectangular region

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- Find $N(T) + 1$ “small” numbers $h_{-1}, \ldots, h_{N(T) - 1}$ such that

$$1 \leq g_{-1} + h_{-1} < \cdots < g_k + h_k < g_{k+1} + h_{k+1} < \cdots < g_{N(T) - 1} + h_{N(T) - 1} \leq T$$

and verify that

$$Z(g_{k-1} + h_{k-1})Z(g_k + h_k) < 0, \quad k = 0, \ldots, N(T) - 1.$$

In case of success all zeros of $\zeta(s)$ with imaginary part between 1 and $T$ do lie exactly on the critical line $\Re(s) = \frac{1}{2}$. 
To summarize. The method recommended is first to find the total number of zeros in the rectangle by methods to be described later. Then to calculate the function at sufficient points to account for all the zeros, either by changes of sign or as complex zeros determined by the use of Rouché’s theorem. We know no way of dealing with multiple zeros, and simply hope that none are present.
To summarize. The method recommended is first to find the total number of zeros in the rectangle by methods to be described later. Then to calculate the function at sufficient points to account for all the zeros, either by changes of sign or as complex zeros determined by the use of Rouché’s theorem. We know no way of dealing with multiple zeros, and simply hope that none are present.

This paper is divided into two parts. The first part is devoted to the analysis connected with the problem. All the results obtained in this part are likely to be applicable to any further calculations to the same end, whether carried out on the Manchester Computer or by any other means. The second part is concerned with the means by which the results were achieved on the Manchester Computer.
**Function $S(t)$**

$N(T)$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1$, $1 \leq \Im(s) \leq T$ should be close to the number of Gram’s points below $T$. 

\[
\int_0^T S(t) \, dt = O(\ln(T))
\]
Function $S(t)$

$N(T)$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1$, $1 \leq \Im(s) \leq T$ should be close to the number of Gram’s points below $T$.

$$g_m < T$$
Function $S(t)$

$N(T)$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1$, $1 \leq \Im(s) \leq T$ should be close to the number of Gram’s points below $T$.

$$g_m < T \iff \theta(g_m) < \theta(T)$$
Function $S(t)$

$N(T)$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1$, $1 \leq \Im(s) \leq T$ should be close to the number of Gram’s points below $T$.

$$g_m < T \iff \theta(g_m) < \theta(T)$$
$$\iff \pi m < \theta(T)$$

Theorem (Littlewood [1924]).

$$\int_0^T S(t) \, dt = O(\ln(T))$$
Function $S(t)$

$N(T)$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1$, $1 \leq \Im(s) \leq T$ should be close to the number of Gram’s points below $T$.

$$g_m < T \iff \theta(g_m) < \theta(T)$$
$$\iff \pi m < \theta(T)$$
$$\iff m < \frac{\theta(T)}{\pi}$$
**Function $S(t)$**

$N(T)$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1, 1 \leq \Im(s) \leq T$ should be close to the number of Gram’s points below $T$.

$$
g_m < T \iff \theta(g_m) < \theta(T)$$

$$
\iff \pi m < \theta(T)
$$

$$
\iff m < \frac{\theta(T)}{\pi}
$$

$$
N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)
$$
**Function $S(t)$**

$N(T)$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1$, $1 \leq \Im(s) \leq T$ should be close to the number of Gram’s points below $T$.

$$g_m < T \iff \theta(g_m) < \theta(T) \iff \pi m < \theta(T) \iff m < \frac{\theta(T)}{\pi}$$

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)$$

**Theorem (Littlewood [1924]).**

$$\int_0^T S(t) \, dt = O(\ln(T))$$
**Function $S(t)$**

$N(T)$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1$, $1 \leq \Im(s) \leq T$ should be close to the number of Gram’s points below $T$.

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)$$

**Theorem (Littlewood [1924]).**

$$\int_0^T S(t) \, dt = O(\ln(T))$$

**Theorem (Turing [1953]).**

If $168 \pi < t_1 < t_2$ then

$$\left| \int_{t_1}^{t_2} S(t) \, dt \right| \leq 2.30 + 0.128 \ln\left(\frac{t_2}{2\pi}\right).$$
**Function \( S(t) \)**

\( N(T) \), the number of zeros of \( \zeta(s) \) inside the rectangular region \( 0 \leq \Re(s) \leq 1, \ 1 \leq \Im(s) \leq T \) should be close to the number of Gram’s points below \( T \).

\[
N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)
\]

**Theorem (Littlewood [1924]).**

\[
\int_{0}^{T} S(t) \, dt = O(\ln(T))
\]

**Theorem (Turing [1953]).** If \( 168\pi < t_1 < t_2 \) then

\[
\left| \int_{t_1}^{t_2} S(t) \, dt \right| \leq 2.30 + 0.128 \ln \left( \frac{t_2}{2\pi} \right).
\]
Calculations of the Riemann zeta-function

Lemma 3.

We have \(|f(0)| < 4\) throughout the strip by the Phragmén–Lindelöf theorem. From this it follows that there are no ordinary maxima separating minima. This equation only has solutions near to \(\epsilon \to 0\), \(R \to \infty\) that the minimum real part must be achieved either on the boundary or in the region \(R > 0\). Also, by Lemma 6,

\[
\int \log \frac{\zeta'(z+2)}{(z+2)^2} \, dz = \int \log \left| \frac{\frac{\zeta'(z+1)}{(z+1)^2}}{\frac{\zeta'(z)}{z^2}} \right| \, dz.
\]

Using Lemma 3 and \(t > 50\), we have

\[
\int \log \frac{\zeta'(z+2)}{(z+2)^2} \, dz = \int \log \left| \frac{\frac{\zeta'(z+1)}{(z+1)^2}}{\frac{\zeta'(z)}{z^2}} \right| \, dz.
\]

Combining these results gives the asserted inequality.
Turing’s Method

\( N(T) \), the number of zeros of \( zeta(s) \) inside the rectangular region \( 0 \leq \Re(s) \leq 1, \ 1 \leq \Im(s) \leq T \) should be close to the number of Gram’s points below \( T \).

\[
N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)
\]
Turing's Method

\( N(T) \), the number of zeros of \( \zeta(s) \) inside the rectangular region
\( 0 \leq \Re(s) \leq 1, \ 1 \leq \Im(s) \leq T \) should be close to the number of Gram’s points below \( T \).

\[
N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)
\]

Let us try to find \( T_0 \) such that \( T < T_0 \) and

\( S(T_0) = 0 \)
Turing’s Method

$N(T)$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1$, $1 \leq \Im(s) \leq T$ should be close to the number of Gram’s points below $T$.

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)$$

Let us try to find $T_0$ such that $T < T_0$ and

$$S(T_0) = 0 \quad \Rightarrow \quad \frac{\theta(T_0)}{\pi} \text{ is an integer}$$
Turing’s Method

\( N(T) \), the number of zeros of \( \zeta(s) \) inside the rectangular region
\( 0 \leq \Re(s) \leq 1, \ 1 \leq \Im(s) \leq T \) should be close to the number of Gram’s points below \( T \).

\[
N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)
\]

Let us try to find \( T_0 \) such that \( T < T_0 \) and

\[
S(T_0) = 0 \quad \Rightarrow \quad \frac{\theta(T_0)}{\pi} \text{ is an integer}
\]

\[
\Rightarrow \quad \sin(\theta(T_0)) = 0
\]
Turing’s Method

$N(T)$, the number of zeros of $\zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1, 1 \leq \Im(s) \leq T$ should be close to the number of Gram’s points below $T$.

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)$$

Let us try to find $T_0$ such that $T < T_0$ and

$$S(T_0) = 0 \Rightarrow \frac{\theta(T_0)}{\pi} \text{ is an integer}$$

$$\Rightarrow \sin(\theta(T_0)) = 0$$

$$\Rightarrow T_0 = g_m$$
Turing’s Method

\[ T_0 = g_m \]
Turing’s Method

\[ T_0 = g_m \quad N(T_0) = N(g_m) = \frac{\theta(g_m)}{\pi} + 1 + S(g_m) \]
Turing’s Method

\[ T_0 = g_m \quad N(T_0) = N(g_m) = \frac{\theta(g_m)}{\pi} + 1 + S(g_m) \]

\[ \sin(\theta(g_m)) = 0 \]
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\[ (-1)^m Z(g_m) > 0 \quad \Rightarrow \quad S(g_m) \text{ is an even integer} \]
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\[
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\Rightarrow \quad \frac{\theta(g_m)}{\pi} \text{ is an integer} \\
\Rightarrow \quad S(g_m) \text{ is an integer}
\]

\[
(-1)^m Z(g_m) > 0 \quad \Rightarrow \quad S(g_m) \text{ is an even integer}
\]

\[
|S(g_m)| < 2 \Rightarrow S(g_m) = 0
\]
Turing’s Method

\[ (-1)^m Z(g_m) > 0 \]
Turing’s Method

\((-1)^m Z(g_m) > 0\)

\((-1)^{m+1} Z(g_{m+1} + h_{m+1}) > 0\)
Turing’s Method

\[ (-1)^m Z(g_m) > 0 \]
\[ (-1)^{m+1} Z(g_{m+1} + h_{m+1}) > 0 \]
\[ \vdots \]
\[ (-1)^{m+k-1} Z(g_{m+k-1} + h_{m+k-1}) > 0 \]
Turing’s Method

\[
(-1)^m Z(g_m) > 0 \\
(-1)^{m+1} Z(g_{m+1} + h_{m+1}) > 0 \\
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(-1)^{m+k} Z(g_{m+k}) > 0
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\[ (-1)^{m+k} Z(g_{m+k}) > 0 \]

Theorem (Turing [1953]).

\[ S(g_m) \leq 1 + \frac{2.30 + 0.128 \ln \left( \frac{g_{m+k}}{2\pi} \right) + \sum_{j=1}^{k-1} h_{m+j}}{g_{m+k} - g_m} \]

\[ S(g_m) \geq -1 - \frac{2.30 + 0.128 \ln \left( \frac{g_m}{2\pi} \right) - \sum_{j=1}^{k-1} h_{m-j}}{g_m - g_{m-k}} \]
PART II. THE COMPUTATIONS

Essentials of the Manchester Computer

It is not intended to give any detailed account of the Manchester Computer here, but a few facts must be mentioned if the strategy of the computation is to be understood. The storage of the machine is of two kinds, known as ‘electronic’ and ‘magnetic’ storage. The electronic storage consisted of four ‘pages’ each of thirty-two lines of forty binary digits. The magnetic storage consisted of a certain number of tracks each of two pages of similar capacity. Only about eight of these tracks were available for the zeta-function calculations. It was possible at any time to transfer one or both pages of a track to the electronic storage by an appropriate instruction. This operation takes about 60 ms. (milliseconds). Transfers to the magnetic store from the electronic were also possible, but were in fact only used for preparatory loading of the magnetic store. The course of the calculations is controlled by instructions each of twenty binary digits. These are normally magnetically stored, but must be transferred to the electronic
The storage available was distributed as follows:

**Magnetic store**
- Logarithms routine (for $\kappa$) . . . . . . . . . . . . 1 page
- Table of logarithms and reciprocal square roots . . . . . . 4 pages
- Routine for calculating the terms $n^{-\frac{1}{2}} \cos 2\pi(\tau \log n - \kappa)$ and table of cosines . . . . . . . . . . . . . 2 pages
- Remainder of routine for calculating the function $Z(\tau)$ . . . . . . 2 pages
- Input routine . . . . . . . . . . . . . . . . . . 2 pages
- Output routine . . . . . . . . . . . . . . . . . . 2 pages

**Electronic store**, as occupied during the greater part of the time
- Instructions and cosines . . . . . . . . . . . . . . . . . . 2 pages
- Logarithms and reciprocal square roots . . . . . . . . . . . . 1 page
- Miscellaneous data and working space . . . . . . . . . . . . 1 page
The results of the calculations are punched out on teleprint tape. This is a slow process in comparison with the calculations, taking about 150 ms. per character. The content of a tape may afterwards automatically be printed out with a typewriter if desired. The significance of what is printed out is determined by the ‘programmer’. In the present case the output consisted mainly of numbers in the scale of 32 using the code

```
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
/ E A S I U D R J N F C K T Z L W
20 21 22 23 24 25 26 27 28 29 30 31
H Y P Q O B G || M X V £
```

and writing the most significant digit on the right. More conventionally the scale of 10 can be used, but this would require the storage of a conversion routine, and the writer was entirely content to see the results in the scale of 32, with which he is sufficiently familiar.
Unfortunately, although the details were all worked out, practically nothing was done on these lines. The interval $1414 < t < 1608$ was investigated and checked, but unfortunately at this point the machine broke down and no further work was done. Furthermore this interval was subsequently found to have been run with a wrong error value, and the most that can consequently be asserted with certainty is that the zeros lie on the critical line up to $t = 1540$, Titchmarsh having investigated as far as 1468 (Titchmarsh (5)).

Unfortunately $0.31E$ was given the inappropriate value of $\frac{1}{128}$ and consequently we are only able to assert the validity of the Riemann hypothesis as far as $t = 1540$, a negligible advance.
### Numerical checking of RH (continued)

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<td>J. B. Rosser, J. M. Yohe, and L. Schoenfeld</td>
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<tr>
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<td>10000000000000000</td>
<td>X. Gourdon</td>
</tr>
</tbody>
</table>
Turing’s papers on Riemann’s Hypothesis

A. M. Turing.

A. M. Turing
Some calculations of the Riemann zeta-function
Turing’s papers on Riemann’s Hypothesis

A. M. Turing.

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A. M Turing.
Systems of logic based on ordinals.
*Proceedings of the London Mathematical Society*
ser. 2, volume 45, 1939, pp. 161–228;

A. M. Turing.
A method for the calculation of the zeta-function.
*Proceedings of the London Mathematical Society*

A. M. Turing
Some calculations of the Riemann zeta-function
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By a number-theoretic theorem we shall mean a theorem of the form \( \theta(x) \) vanishes for infinitely many natural numbers \( x \), where \( \theta(x) \) is a primitive recursive function.
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\(^5\)I believe that there is no generally accepted meaning for this term, but it should be noticed that we are using it in a rather restricted sense.
Systems of logic based on ordinals.


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By a number-theoretic theorem\(^5\) we shall mean a theorem of the form “\(\theta(x)\) vanishes for infinitely many natural numbers \(x\)”, where \(\theta(x)\) is a primitive recursive function.

An alternative form for number-theoretic theorems is ”for each natural number \(x\) there exists a natural number \(y\) such that \(f(x, y)\) vanishes”, where \(f(x, y)\) is primitive recursive.

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RH in the Arithmetical Hierarchy

\[ \Pi_0^0 = \Sigma_0^0 = \{ \phi \mid \text{all quantifiers in } \phi \text{ are bounded} \} \]
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\[ \Pi^0_2 = \{ \forall x_1 \ldots x_m \phi \mid \phi \in \Sigma^0_1 \} \]
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\vdots \quad \vdots
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\[ \begin{align*}
\Pi_1^0 &= \{ \forall x_1 \ldots x_m \phi \mid \phi \in \Sigma_0^0 \} \\
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\vdots & \\
\Pi_{n+1}^0 &= \{ \forall x_1 \ldots x_m \phi \mid \phi \in \Sigma_n^0 \} \\
\end{align*} \]

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Where does RH lie in this hierarchy?
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Where does RH lie in this hierarchy?

RH ∈ \( \Pi_0^0 = \Sigma_0^0 \)

Either \( RH \iff 0 = 0 \) or \( RH \iff 0 = 1 \)
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\[ \vdots \quad \vdots \]

\[ \Pi^0_{n+1} = \{ \forall x_1 \ldots x_m \phi \mid \phi \in \Sigma^0_n \} \quad \Sigma^0_{n+1} = \{ \exists x_1 \ldots x_m \phi \mid \phi \in \Pi^0_n \} \]

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Given what we know today, where in this hierarchy can we find a formula equivalent to RH?
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**Theorem (Kreisel [1958]).** \( \text{RH} \in \Pi_1^0 = \{ \forall x_1 \ldots x_m \phi \mid \phi \in \Sigma_0^0 \} \).
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**Theorem (Kreisel [1958]).** RH \(\in\) \(\Pi_1^0 = \{ \forall x_1 \ldots x_m \phi \mid \phi \in \Sigma_0^0 \} \).

**Corollary of DPRM-Theorem [1970].** RH \(\iff \forall x_1 \ldots x_m P(x_1, \ldots, x_m) \neq 0 \).