Solving the Facility Location Problem Using Message Passing

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Facility Location

Facility Location (FL) Problem:
Open a subset of facilities & connect customers to one facility each at minimal cost
FL in Machine Learning

- Exemplar-based clustering
  - $C = F$

- Multiple model selection [Li`07]
  - $F$: models
  - $C$: data

- Many practical problems…
  - wireless sensor networks
  - computational biology
  - computer vision
Outline

- Approach to FL:
  - Approximate MAP inference in graphical model
  - Max-product linear programming (MPLP) [Globerson & Jaakkola ’08]

- MPLP fixed points
  - Unique solution: guaranteed optimal
  - Non-unique solution: unknown how to set some variables

- Today
  - New greedy algorithm for decoding variables for FL
  - In some cases, does not coincide with any MPLP variable assignment
  - Optimality guarantees (3-approximation)
  - Empirically better solutions than typical MPLP solutions
\[ \min_x \sum_{ij} c_{ij} x_{ij} + \sum_j f_j \max_i x_{ij} \]

s.t. \[ \sum_j x_{ij} = 1 \]

\( x_{ij} \): customer \( i \) connected to facility \( j \)

1-of-N binary encoding

\( c_{ij} \): connection costs

\( f_j \): facility costs
Background: MPLP

MAP: \[ \min_x \sum_c \theta_c(x_c) \]

MAP-LP: \[ \min_{\mu} \sum_c \sum_{x_c} \mu_c(x_c) \theta_c(x_c) \]

Lower bound on MAP.

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<tr>
<th>$x_1x_2$</th>
<th>$\mu_a(x_1,x_2)$</th>
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<tr>
<th>$x_2x_3$</th>
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</table>
Background: MPLP

MAP: \( \min_x \sum_c \theta_c(x_c) \)

MAP-LP: \( \min_\mu \sum_c \sum_{x_c} \mu_c(x_c) \theta_c(x_c) \)

Lower bound on MAP.

\( x \equiv \text{integral } \mu. \)

\( x = 011 \)
Background: MPLP

Dual LP: sum of beliefs
\[ \sum_i \max_{x_i} b_i(x_i) \]
where \( b_i(x_i) = \sum_c m_{ci}(x_i) \)

MPLP:
- Iteratively update \( m_{ci}(x_i) \)
- Compute beliefs: \( b_i(x_i) = \sum_c m_{ci}(x_i) \)
- Assign variables: \( x_i^* = \arg \max_{x_i} b_i(x_i) \)
MPLP fixed points

Two cases:

- $x^*$ unique - optimal solution
- $b_i(1)=b_i(0)$ for some variables
  - Can find optimal in special cases (e.g. binary $x$, pairwise submodular $\theta$)
  - Optimal unknown if $NP$-hard

$x_i^* = \arg \max_{x_i} b_i(x_i)$
Complementary slackness

- Our approach: **complementary slackness conditions**
  - Always hold for a pair of LP solutions \((\mu^*, \beta^*)\) that are **primal** and **dual** optimal

- MPLP: \(x^* = \arg \max_x b(x)\)
  - \(\mu^* = x^*\) satisfies a **subset** of c.s. conditions for MAP LP
  - Greedily try satisfy **all** c.s. conditions & achieve the LP lower bound
FL: complementary slackness

Solution support graph $G=(C,F,E)$:

MPLP fixed point:
edges $b_{ij} \geq 0$

Integral solution $x$:
edges $x_{ij} = 1$

$b_{ij} \equiv b_{ij}(1) - b_{ij}(0)$
Complementary slackness for $x$:

1. Customers: connected via an -- edge
2. Facilities: all or no -- edges

Diagram:

- $b_{ij} \geq 0$
- $x_{ij} = 1$
FL: complementary slackness

Complementary slackness for $x$:
1. Customers: connected via an edge
2. Facilities: all or no edges

$1$ violated.
Complementary slackness for $x$:
1. Customers: connected via an -- edge
2. Facilities: all or no -- edges

$2$ violated.
Complementary slackness for $x$:
1. Customers: connected via an -- edge
2. Facilities: all or no -- edges

LPr tight – can satisfy both 1 and 2.
FL: complementary slackness

Complementary slackness for $\mathbf{x}$:

1. Customers: connected via an -- edge
2. Facilities: all or no -- edges

LPr tight – can satisfy both 1 and 2.
Decoding: belief maximization

\[ x_{ij}^* = \arg \max_{x_{ij}} b_{ij}(x_{ij}) \quad - \text{always satisfies c.s. 1} \]

- pick an edge for each customer
Decoding: belief maximization

\[ x_{ij}^* = \arg \max_{x_{ij}} b_{ij}(x_{ij}) \]  
- always satisfies c.s. 1
- pick an edge for each customer

One possible solution – all facilities are open!
Decoding: greedy algorithm

Our approach:
- always satisfy c.s. 2
- open facilities in a greedy order
Decoding: greedy algorithm

Our approach:
- always satisfy c.s. 2
- open facilities in a greedy order

1 violated - $x_{ij}$ does not maximize its belief
FL Approximability

$\rho$-approximation algorithm: guarantees $E(x^*) \leq \rho E(x^{OPT})$
- $\rho$ is $O(\ln|C|)$ in general
- $\rho$ is constant for *metric* FL

$c_{ij} \leq c_{ik} + c_{lk} + c_{lj}$
3-approximation for metric FL

- Integral ≤ 3 Dual
- Proof: triangle inequality, greedy order, fixed point

- Integral = Dual
- Proof: complementary slackness

Integral ≤ 3 Dual ≤ 3 Optimal
Experimental evaluation

- Metric FL, $C=F$
  - uniformly sampled 2D points
  - $c_{ij}$: Euclidean distance
  - $f_j$: equal for all $j$

- Algorithms
  - MPLP + Beliefs
  - MPLP + Greedy

- Error: % above LP lower bound
Experimental evaluation

- Metric FL, $C=F$

- Standard max-product: Affinity Propagation (AP)

- Algorithms
  - AP + Beliefs
  - AP + Greedy
  - MPLP + Beliefs
  - MPLP + Greedy
Experimental evaluation

- Non-metric ORLIB benchmarks
- Algorithms
  - MP + Beliefs
  - MP + Greedy
  - MPLP + Beliefs
  - MPLP + Greedy
- Error: % above optimal

| Name  | |C| | |F| | |Opt.|
|-------|---|---|---|---|---|
| c7*   | 50 | 16 | 4/4 |
| c10*  | 50 | 25 | 4/4 |
| c13*  | 50 | 50 | 4/4 |
| a,b,c | 1000 | 100 | 1/3 |
Summary and conclusions

- Facility Location
  - Graphical model + MPLP + Greedy decoding
  - 3-approximation for metric FL
  - Improved empirical results over maximizing beliefs

- Questions?