Superconductivity: Electron-phonon Coupling and Unconventional Pairing with Repulsive Interaction

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MEPhI was founded in 1942 as the Moscow Mechanical Institute of Munitions

1 Historical overview
2 Some theoretical concepts and e-ph coupling.
3 Superconductivity with repulsion.
Heike Kamerlingh Onnes (21 September 1853 – 21 February 1926) was a Dutch physicist. His scientific career was spent exploring extremely cold refrigeration techniques and the associated phenomena.

- 1908 he was the first physicist to liquify helium. Reducing the pressure of helium allows to reach 0.9K. *Birthday of low temperature physics.*
- 1911 The resistivity of mercury drops unexpectedly to zero at 4.2K. *Birthday of superconductivity.*
- 1913 Nobel prize in physics.
- In the field of superconductivity and superfluidity it was awarded 7 Nobel prices, 6 of them in the last 40 years (the last in 2003).
Few historical facts

1. Problem of obtaining the required amount of helium gas. They manage to get 360 liters of helium gas from the sand from North Carolina.

2. Measurements of resistivity of pure metals at low T. First platinum and gold. Then mercury was chosen because of easy purification.

3. Zero resistance has been found. But it was attributed to a short circuit somewhere in the cryostat.
Because of deficit of helium the pressure in the cryostat was low, to ensure that helium does not escape from the cryostat.

Boiling point of He is 4.2 K. Critical temperature of mercury is 4.15K

The finite resistivity was found, because student was not able to keep the pressure low!

Very soon it was discovered superconductivity in tin and lead. Then it was discovered, that external magnetic field as well as superconducting current suppresses superconductivity. It was attempts to measure isotope effect and Meissner -Ochsenfeld effect(1933).

“There is no doubt that very pure gold and platinum are superconducting at low temperatures.”
London Equations

\[ \Lambda \nabla \times J_s = -H / c, \quad \Lambda = \frac{m}{e^2 n_s} \]

The meaning of London theory:

\[ \frac{\partial V_s}{\partial t} = - (V_s \nabla) V_s + \frac{e}{m} E + \frac{e}{mc} V_s \times H \equiv \]

\[ \frac{e}{m} E - \nabla \frac{V_s^2}{2} + V_s \times (\nabla \times V_s + \frac{e}{mc} H) \]

\[ \lambda^2 = \frac{mc^2}{4 \pi e^2 n_s} \quad \text{London penetration depth} \]
The Nobel Prize in Physics 2003

"for pioneering contributions to the theory of superconductors and superfluids"

**Alexei A. Abrikosov**  
1/3 of the prize  
USA and Russia  
Argonne National Laboratory  
Argonne, IL, USA

**Vitaly L. Ginzburg**  
1/3 of the prize  
Russia  
P.N. Lebedev Physical Institute  
Moscow, Russia

**Anthony J. Leggett**  
1/3 of the prize  
United Kingdom and USA  
University of Illinois Urbana, IL, USA
Ginzburg-Landau Theory

(Nobel prize 2003).

\( \Psi(r) \)  
Order parameter – scalar complex function

\[
F_s = F_0 + \frac{H^2}{8\pi} + \frac{1}{4m} \left| -i\hbar \nabla \Psi - \frac{2e}{c} A \Psi \right|^2 + \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4
\]

\[
\frac{1}{4m} \left( -i\hbar \nabla - \frac{2e}{c} A \right)^2 |\Psi + \alpha \Psi + \frac{\beta}{2} |\Psi|^2 \Psi = 0
\]

\[
\mathbf{j}_s = -\frac{ie\hbar}{2m} \left( \Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right) - \frac{2e}{mc} |\Psi|^2 \mathbf{A}
\]

! Interesting story about charge 2e in GL theory!
Magnetic field penetrates to the superconductor in the form of tubes. Outside of this tubes field is absent.

\[ \xi^2 = \frac{\hbar^2}{4m|\alpha|} \]  
-Coherence length

\[ \lambda^2 = \frac{mc\beta}{8\pi e^2 |\alpha|} \]  
-Penetration depth

\[ \kappa = \frac{\lambda}{\xi} < \frac{1}{\sqrt{2}} \]  
-Type I superconductors

\[ \kappa > \frac{1}{\sqrt{2}} \]  
-Type II superconductors

Vortices in in type II superconductors Abrikosov phase
Microscopic theory of superconductivity (1956)

The Nobel Prize in Physics 1972

"for their jointly developed theory of superconductivity, usually called the BCS-theory"

John Bardeen
1/3 of the prize
USA
University of Illinois Urbana, IL, USA

Leon Neil Cooper
1/3 of the prize
USA
Brown University Providence, RI, USA

John Robert Schrieffer
1/3 of the prize
USA
University of Pennsylvania Philadelphia, PA, USA
Superconducting transition is the phase transition of the second order which is described by the complex order parameter:

\[ \Delta_{s_1 s_2} (\vec{r}_1, \vec{r}_2) \propto \left\langle \psi_{s_1} (\vec{r}_1) \psi_{s_2} (\vec{r}_2) \right\rangle \quad \Delta_{s_1 s_2} (\vec{R}, \vec{r}) = -\Delta_{s_2 s_1} (\vec{R}, -\vec{r}) \]

Elementary excitations have gapped spectrum.

The ground state is described by the order parameter which have the meaning of the quantum mechanical wave function.

The properties of the system at large energies are not changed with respect to that of the normal state.

Critical temperature is determined by the interactions in the normal metallic state, i.e. by Coulomb pseudopotential and coupling to phonons.
Discovery of high temperature superconductivity (1986)

The Nobel Prize in Physics 1987

"for their important break-through in the discovery of superconductivity in ceramic materials"

J. Georg Bednorz
1/2 of the prize
Federal Republic of Germany
IBM Zurich Research Laboratory
Rüschlikon, Switzerland

K. Alexander Müller
1/2 of the prize
Switzerland
IBM Zurich Research Laboratory
Rüschlikon, Switzerland
Eliashberg theory of superconductivity

Coulomb pseudopotential:

\[ \mu_c = V(q \approx 0)N(0) \]

This parameter is dimensionless and tells us if the correlations are weak or strong in the metal.

Coulomb pseudopotential is renormalized due to retardation effect

Electron-phonon interaction in metals

\[ H_{e-ph} = \sum_{k,q,\alpha} g(k, q) u_{q,\alpha} c_{k+q}^\dagger c_k \]

Isotropic case, nothing is momentum dependent:

\[ Q(\omega, \xi, \xi') = \frac{1}{\hbar N(0)} \sum_{k,q} g(k, q)^2 \delta(\omega - \omega_q) \delta(E_k - E_F - \xi) \delta(E_{k+q} - E_F - \xi') \]
Electron-phonon interaction in superconductors

\[ Q(\omega, \xi, \xi') \]
is a complicated function

\[ E_k, E_{k+q} \approx E_F >> \omega_q \approx \omega_D \quad \Rightarrow \quad Q(\omega, \xi, \xi') \approx Q(\omega, 0, 0) \equiv \alpha^2 F(\omega) \]
is a spectral function of electron-phonon interaction or Eliashberg function. This function describes electron-phonon interaction in superconductors.

\[ \lambda \langle \omega^n \rangle = 2 \int_0^\infty d\omega \frac{\alpha^2 F(\omega) \omega^n}{\omega} \]
\(n\)-th moment of Eliashberg function

\[ \lambda = 2 \int_0^\infty d\omega \frac{\alpha^2 F(\omega)}{\omega} \]
Dimensionless electron-phonon coupling constant
Superconducting metals

There are two dimensionless parameters to characterize superconducting metals:

\[ \lambda = 2 \int_{0}^{\infty} d\omega \frac{\alpha^2 F(\omega)}{\omega} \]

Electron-phonon coupling constant

\[ \mu_c^* = \frac{\mu_c}{\left[1 + \mu_c \ln\left(\frac{E_F}{\omega_{ph}}\right)\right]} \]

Coulomb pseudopotential, reduced due to retardation effects

\[ \Delta \rightarrow \Delta(\omega) \]
Kinetics of hot electrons in the normal states.
Pump-probe spectroscopy.

\[ \frac{\Delta R}{R}(t); \quad \frac{\Delta T}{T}(t); \]
\[ \tau_i(T, P), A_i(T, P) \]
\[ \mu_c; \quad \lambda \left\langle \omega^n \right\rangle \]
Electron-Phonon Coupling in High-Temperature Cuprate Superconductors
Determined from Electron Relaxation Rates

C. Gadermaier, A. S. Alexandrov, V. V. Kabanov, P. Kusar, T. Mertelj, X. Yao, C. Manzoni, D. Brida, G. Cerullo, and D. Mihailovic

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Background

NEW MECHANISM FOR SUPERCONDUCTIVITY*

W. Kohn

University of California, San Diego, La Jolla, California

and

J. M. Luttinger

Columbia University, New York, New York
(Received 16 August 1965)

For electrons in metals

\[ u(\mathbf{r}) = +(4\pi a/m)\delta(\mathbf{r}), \]

\[ u(0) = \frac{\pi^2}{m k_F} \]

\[ \left( \frac{k T_c}{\varepsilon_0}, l \right) \sim e^{-40} \sim 10^{-17}, \]

\[ \left( \frac{k T_c}{\varepsilon_0} \right) \sim \exp[-(2l)^4]. \]

Secondly, by taking metals with different parameters and nonspherical Fermi surfaces, it may prove possible to enhance the effect appreciably. In fact, it is known that some
Order parameter

\[ \Delta_{s_1s_2} (\vec{r}_1, \vec{r}_2) = \Delta_{s_1s_2} (\vec{R}, \vec{r}) = -\Delta_{s_2s_1} (\vec{R}, -\vec{r}) \]

In our simple case we consider uniform case i.e

\[ \Delta_{s_1s_2} (\vec{R}, \vec{r}) = \Delta_{s_1s_2} (\vec{r}) \Rightarrow \Delta(\vec{q}) \]
More details

\[
\Delta(k) = \int d\xi \frac{\tanh(\xi / 2T_c)}{2\xi} \frac{\Omega}{(2\pi\hbar)^d} \sum_{k'} K(k, k')\Delta(k')
\]

Equation for \( \lambda \)

\[
\lambda \Delta(p) = \frac{\Omega}{(2\pi\hbar)^d} \int \frac{dS}{v_F(S)} K(p, p')\Delta(p')
\]

\[
K(p, p') = \nu(p - p') + \nu(p - p') \sum_k [2\nu(p - p')
- \nu(k + p') - \nu(k - p)]Q(p - p', k)
- \sum_k \nu(p - k)\nu(k + p')Q(p + p', k)
\]

\[
Q(q, k) = (f_k - f_{k-q})/(E_k - E_{k-q})
\]
This diagrams are very difficult to evaluate if repulsion $V(q)$ is $q$-dependent!
Therefore most of the previous and recent studies confined to hard sphere (Hubbard U) repulsion $V(q)=\text{const}$!

Diagrams b), c), d) cancel each other. Diagram e) is known Lindhard function:

\[
\Gamma_a \propto U(0) ; \quad \Gamma_e \propto U(0)^2 \chi(q) = U(0)^2 \left[ 1 + \frac{k_F^2 - q^2}{4 q k_F} \ln \left( \frac{k_F + q/2}{k_F - q/2} \right) \right] / 2
\]

\[
\lambda_l = \int_0^\pi d\theta \sin \theta P_l(\cos \theta) (\Gamma_a + \Gamma_e)
\]
Hubbard model: \[ \Gamma_a = \text{const} \quad \Rightarrow \quad \lambda_a \propto \delta_{l0} \]

\[ \lambda_{le} \propto 1/l^4 \]

Weakly interacting Hubbard model is superconducting in 3D

\[ V(q) = \frac{4\pi e^2}{q^2 + \kappa^2} \Rightarrow U = V(0) = \frac{\pi^2}{mk_F} \]

\[ \lambda_1 = \frac{2 \ln 2 - 1}{40} = 0.0096 \]
\[ \lambda_2 = \frac{8 - 11 \ln 2}{420} = 0.00089 \]
In 2D the screening is different and the pairing is possible only because of nontrivial Fermi surface.

The exchange diagram e) is given by:

\[
\chi(q) = \frac{m}{2\pi} \left[ 1 - \frac{\text{Re}\sqrt{q^2 - (2k_F)^2}}{q} \right].
\]

Pairing is absent because it does not have negative contributions for any \( l \) ! To have superconductivity in 2D it is important to take into account nonspherical shape of Fermi surface.

Only diagrams a) and e) are not equal to 0!
Superconductivity from repulsive interactions in the two-dimensional electron gas

S. Raghu$^{1,2}$ and S. A. Kivelson$^1$

In the small $r_s$ limit, the Coulomb interactions are sufficiently well screened that it may be reasonable to treat them as weak and short-ranged.

\[
V(q) = \frac{4\pi e^2}{q^2 + \kappa^2(q)} \quad \text{where} \quad \kappa^2(q) = 2k_F^2 r_s \chi(q) / 3 \ll k_F^2!
\]

\[
r_s = 1.92e^2 / \hbar v_F < 1
\]

Therefore screening radius is large in comparison with the distance between particles. q-dependence is important in metals with small $r_s \ll 1$

Note that the line from my paper with Raghu, ("in the small rs limit, the Coulomb interactions are sufficiently well screened that it may be reasonable to treat them as weak and short-ranged") contains the carefully inserted modifier, "may," as this is not something we could establish.
Unconventional High-Temperature Superconductivity from Repulsive Interactions: Theoretical Constraints

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Where superconductivity is possible?

\[ V(q) = \frac{4\pi e^2}{q^2 + \kappa^2}, \kappa >> k_F \Rightarrow V(q) = \frac{4\pi e^2}{\kappa^2} \left[ 1 - \frac{q^2}{\kappa^2} + \frac{q^4}{\kappa^4} + \ldots \right] \]

\[ \lambda_1 / s = \left( \frac{k_F}{\kappa} \right)^4 \left( \frac{3}{4} - 2s(2 \ln 2 - 1) / 5 \right) \Rightarrow r_s > 36! \]

\[ \lambda_2 / s = \left( \frac{k_F}{\kappa} \right)^4 \left( 16k_F^2 / \kappa^2 15 - 4s(8 - 11\ln 2) / 105 \right) \Rightarrow \left( \frac{\kappa}{k_F} \right)^2 > 450 / r_s! \]

P-wave superconductivity is outside of applicability of the theory, d-wave superconductivity requires screening radius which is at least 1 order of magnitude shorter than lattice constant!
Self-consistent calculations

\[
\left( \frac{\kappa}{k_F} \right)^2 \rightarrow s(q) = 4s \left[ \frac{1}{2} + \frac{k_F^2 - q^2/4}{2qk_F} \ln \frac{k_F}{k_F - q/2} \right].
\]

\[
(\lambda_3 \approx 0.0011 \text{ for } s = 3)
\]

\[
T_c \approx (E_F/k_B) \exp(-1/\lambda)
\]

\[
s = r_s / 6
\]
Conclusions

Thank you for the attention