Phase transition in the family of $p$-resistances

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**Resistance distance** $R(s, t)$

Consider the electrical network corresponding to a graph.

- **R(s,t)**: The effective resistance between $s$ and $t$.

$$R(s, t) = \min_i \sum_{e \in E} r_e i_e^2 \quad i = (i_e)_{e \in E} \text{ is a unit } s-t \text{ flow.}$$

**Pro:** In small graphs, it captures the cluster structure!

- Con: (von Luxburg et al. 2010) In large geometric graphs, it converges to the trivial limit

$$R(s, t) \approx \frac{1}{d_s} + \frac{1}{d_t}$$
How we can cure this problem?

\[ p \text{-Resistance}: \text{For } p \geq 1, \text{ define} \]

\[ R_p(s, t) := \min_i \sum_{e \text{ edge}} r_e |i_e|^p \]

**Theorem (Special cases of } R_p(s, t)\text{)**

- **\( p = 1 \)**: Shortest path distance
- **\( p = 2 \)**: Standard resistance distance
- **\( p \to \infty \)**: Related to \( s-t \)-mincut

\[ p = 2 \]

\[ p = 1.33 \]

\[ p = 1.1 \]
Main Theorem:

For **large** random geometric graphs in $R^d$:

1. If $p < 1 + 1/(d - 1)$, then the “global” contribution dominates the “local” one.
   \[ \sim \text{meaningful distance} \]

2. If $p > 1 + 1/(d - 2)$, then all “global” information vanishes.
   \[ \sim \text{useless distance} \]