Sample complexity bounds for differentially private learning

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Outline

1. Learning and privacy model
2. Our results: sample complexity bounds for differentially-private learning
3. Recap & future work
Part 1. Learning and privacy model
Data analytics with sensitive information

**eCommerce:** customers’ browsing & purchase histories

**Clinical studies:** patients’ medical records & test results

**Genomic studies:** subjects’ genetic sequences

Learn something useful about *whole population* from *data about individuals.*
Data analytics with sensitive information

**eCommerce**: customers’ browsing & purchase histories

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**Genomic studies**: subjects’ genetic sequences

Learn something useful about whole population from data about individuals.

This work: learning a binary classifier from labeled examples, where each training example is an individual’s sensitive information.
Data analytics with sensitive information

Sensitive training data → Learning algorithm → Publicly-released classifier
Q: If a classifier is learned from some individuals’ sensitive data, can releasing / deploying the classifier in public violate the privacy of individuals from the training data?
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A: Yes! Even after standard “anonymization”, and even when just releasing aggregate statistics, because an adversary could have side-information.
Example: genome-wide association studies

Wang et al (2009): able to combine side-information and published correlation statistics to determine whether an individual from the study was in disease group or healthy group.
Goals: learn an accurate classifier from sensitive data while also preserving the privacy of the data.

This work: how many labeled examples are needed to achieve both of these goals simultaneously?
Goal 1: Differential privacy

What kind of privacy guarantee can a good learning algorithm provide?

Differential privacy guarantee [Dwork et al, 2006]: an individual’s inclusion in the training data does not change (much) what an adversary could learn about that individual’s sensitive information.
**Goal 1: Differential privacy**

(Definition from [Dwork, et al 2006], specialized to learning [Kasiviswanathan, et al 2008])

A learning algorithm $A: (\mathcal{X} \times \{0, 1\})^* \rightarrow \mathcal{H}$ is $\alpha$-differentially private if:

For all training sets $S$ and $S'$ differing in at most one example,

$$\forall G \subseteq \mathcal{H}, \quad \frac{\Pr_A[A(S) \in G]}{\Pr_A[A(S') \in G]} \leq e^\alpha.$$

- Probability is over internal randomness of the learning algorithm.
- Algorithm must behave similarly given similar training sets.
- Smaller $\alpha \in [0, 1]$ corresponds to stronger guarantee.
Goal 2: Learning

Standard statistical learning guarantees:

If $S$ is an i.i.d. sample from a distribution $\mathcal{P}$ over $\mathcal{X} \times \{0, 1\}$, then $\mathcal{A}(S)$ returns a hypothesis $h \in \mathcal{H}$ such that w.p. $\geq 1 - \delta$ (over random draw of $S$ and randomness in $\mathcal{A}$)

$$
\text{err}_\mathcal{P}(h) \leq \min_{h' \in \mathcal{H}} \text{err}_\mathcal{P}(h') + \epsilon
$$

where $\text{err}_\mathcal{P}(\tilde{h}) = \Pr_{(x,y) \sim \mathcal{P}}[\tilde{h}(x) \neq y]$. 
What was known
(previous work)

• Sample complexity for finite hypothesis classes or VC classes over discrete data domains.

\[ C \cdot \left( \frac{1}{\alpha \epsilon} + \frac{1}{\epsilon^2} \right) \cdot \left( \min\{\log |\mathcal{H}|, \ VC_{\mathcal{H}} \log |\mathcal{X}|\} + \log \frac{1}{\delta} \right) \]

• Related problems: (synthetic) data set release.
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- Related problems: (synthetic) data set release.

What about infinite classes & continuous data domains?
Part 2. Sample complexity bounds for differentially-private learning
Our results

1. **Some bad news**: no distribution-independent sample complexity upper bound possible for differentially-private learning.

2. **Some hope**: differentially-private learning possible if
   a. learner allowed some prior-knowledge, or
   b. privacy requirement is relaxed.
I. No distribution-independent sample complexity upper bound

Let $\mathcal{H}$ be the class of threshold functions on the unit interval $[0, 1]$, and pick any positive real number $M$.

For every $\alpha$-differentially private algorithm $A: ([0, 1] \times \{0, 1\})^* \rightarrow \mathcal{H}$, there is a distribution $\mathcal{P}$ (with full support) over $[0, 1] \times \{0, 1\}$ such that:

1. There exists a threshold $h^* \in \mathcal{H}$ with $\text{err}_\mathcal{P}(h^*) = 0$.

2. If $S$ is an i.i.d. sample of size $m \leq M$ from $\mathcal{P}$, then

$$\Pr_{S \sim \mathcal{P}^m, A} \left[ \text{err}_\mathcal{P}(A(S)) > \frac{1}{5} \right] \geq \frac{1}{2}.$$
1. No distribution-independent sample complexity upper bound

Implications:

1. No direct analogue of VC theorem for differentially-private learning.

2. Qualitative difference between finite hypothesis class / discrete data domains and infinite classes / continuous data domains.
I. No distribution-independent sample complexity upper bound

**Proof idea:** find data distributions $P$ and $P'$ such that a “successful” distribution over thresholds for $P$ differs significantly from a “successful” distribution over thresholds for $P'$.

A differentially-private learner using just a small number of examples must behave similarly in both cases; therefore, it must fail for at least one of the cases.
2. Some hope for differentially-private learning

Possible ways around the lower-bound:

a. Allow learner access to prior-knowledge (or prior belief) about unlabeled data distribution.

b. Only guarantee the differential privacy of the labels in the training data.
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Possible ways around the lower-bound:

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b. Only guarantee the differential privacy of the labels in the training data.
2(a). Upper bounds based on prior knowledge of unlabeled data distribution

- Allow learner access to a *reference distribution* $U$ over unlabeled data $X$, chosen independently of the training data.

- Sample complexity upper bound depends on how close $U$ is to $D$ (true unlabeled data distribution).

\[ U \text{ and } D \text{ close} \quad U \text{ and } D \text{ far} \]
2(a). Upper bounds based on prior knowledge of unlabeled data distribution

Let \( \mathcal{P} \) be any distribution over \( \mathcal{X} \times \{0, 1\} \) with marginal \( \mathcal{D} \) over \( \mathcal{X} \). There is a constant \( C > 0 \) and an \( \alpha \)-differentially private algorithm \( \mathcal{A}_1 \) s.t. given an i.i.d. sample \( S \) of size

\[
|S| \geq C \cdot \left( \frac{1}{\alpha \varepsilon} + \frac{1}{\varepsilon^2} \right) \cdot \left( d_\mathcal{U} \cdot \log \frac{\kappa(\mathcal{U}, \mathcal{D})}{\varepsilon} + \log \frac{1}{\delta} \right),
\]

w.p. \( \geq 1 - \delta \), \( \mathcal{A}_1(S) \) returns a hypothesis \( h \in \mathcal{H} \) with \( \text{err}_\mathcal{P}(h) \leq \min_{h' \in \mathcal{H}} \text{err}_\mathcal{P}(h') + \varepsilon \).

\( d_\mathcal{U} \): doubling-dimension of disagreement metric w.r.t. \( \mathcal{U} \).
\( \kappa(\mathcal{U}, \mathcal{D}) \): divergence measure between distributions \( \mathcal{U} \) and \( \mathcal{D} \).
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2(a). Upper bounds based on prior knowledge of unlabeled data distribution

Let $\mathcal{P}$ be any distribution over $\mathcal{X} \times \{0, 1\}$ with marginal $\mathcal{D}$ over $\mathcal{X}$. There is a constant $C > 0$ and an $\alpha$-differentially private algorithm $A_1$ s.t. given an i.i.d. sample $S$ of size

$$|S| \geq C \cdot \left( \frac{1}{\alpha \epsilon} + \frac{1}{\epsilon^2} \right) \cdot \left( d_{\mathcal{U}} \cdot \log \frac{\kappa(\mathcal{U}, \mathcal{D})}{\epsilon} + \log \frac{1}{\delta} \right),$$

w.p. $\geq 1 - \delta$, $A_1(S)$ returns a hypothesis $h \in \mathcal{H}$ with $\text{err}_{\mathcal{P}}(h) \leq \min_{h' \in \mathcal{H}} \text{err}_{\mathcal{P}}(h') + \epsilon$.

$d_{\mathcal{U}}$: doubling-dimension of disagreement metric w.r.t. $\mathcal{U}$.

$\kappa(\mathcal{U}, \mathcal{D})$: divergence measure between distributions $\mathcal{U}$ and $\mathcal{D}$. 
2(a). Upper bounds based on prior knowledge of unlabeled data distribution

Example:

- \(H = n\)-dimensional linear separators through the origin
- \(U = \) uniform distribution on unit sphere (so \(d_U = O(n)\))
- Unlabeled data distribution \(D\) close to uniform: \(D(x) \leq c \cdot U(x)\)
- Sample complexity upper bound:
  \[C \cdot \left(\frac{1}{\alpha \epsilon} + \frac{1}{\epsilon^2}\right) \cdot \left(n \cdot \log \frac{c}{\epsilon} + \log \frac{1}{\delta}\right)\]
Recap & future work


2. Some ways out:
   a. Data-dependent bounds based on prior-knowledge.
   b. Relaxed notion of privacy (label privacy).

3. Future directions:
   a. Improper learning (some work in discrete settings by [Beimel et al, 2010]).
   b. Other weaker notions of privacy.
   c. More general statistical estimation tasks.
Thanks!
2(a). Upper bounds based on *prior knowledge* of unlabeled data distribution

Example:

- $H = n$-dimensional linear separators through the origin
- $U = \text{uniform distribution on unit sphere}$ (so $d_U = n$)
- Unlabeled data distribution $D$ uniform outside $\Theta(1)$-width band around equator.
- Sample complexity upper bound:

\[
C \cdot \left( \frac{1}{\alpha \epsilon} + \frac{1}{\epsilon^2} \right) \cdot \left( n^2 + n \cdot \log \frac{1}{\epsilon} + \log \frac{1}{\delta} \right)
\]
1. Bad news: no distribution-independent sample complexity upper bound

Idea: Consider a set of distributions \( \{ P_z \} \) for \( z \in [0,1] \): the marginal of each \( P_z \) over \( X \) is an even mixture of

1. uniform on \([0,1]\), and
2. uniform on \([z-\eta, z+\eta]\) \( (\text{where } \eta = \Theta(\exp(-\alpha M))) \);

and labels are given by threshold \( h_z(x) = 1[x \geq z] \).

To show: Every \( \alpha \)-differentially private learning algorithm using at most \( M \) training examples will fail on at least one distribution \( P_z \).
1. Bad news: no distribution-independent sample complexity upper bound

A “successful” distribution over thresholds for $P_z$ differs significantly from a “successful” distribution over thresholds for $P_{z'}$.

However, a differentially-private learner using a small number of examples must behave similarly in both cases.
2(b). Label privacy

- Weaker privacy guarantee: only guarantee differential-privacy of the labels.
- Can still protect against some privacy attacks on training data.

A learning algorithm $A : (\mathcal{X} \times \{0, 1\})^* \rightarrow \mathcal{H}$ is $\alpha$-label private if:

For all training sets $S, S' \subseteq \mathcal{X} \times \{0, 1\}$ differing in at most one label,

$$\Pr_A[A(S) \in G] \leq \Pr_A[A(S') \in G] \cdot e^{\alpha} \quad (\forall G \subseteq \mathcal{H})$$
2(b). Label privacy

• Label privacy avoids complications that arise with infinite hypothesis classes and continuous data domains.

• Can obtain upper- and lower-bounds in terms of certain distribution-dependent complexity measures (covering number, doubling dimension).

• Bounds are (roughly) within $1/\alpha$ factor of non-private sample complexity bounds.
Goal 1: Privacy (?)

What kind of privacy guarantee can a good learning algorithm provide?
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Possible guarantee: an adversary does not learn new information about an individual’s sensitive information from the released classifier $h$. 
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Not possible: e.g., adversary who knows person $i$’s feature vector can accurately predict person $i$’s label with $h$!
Doubling dimension

- Hypothesis class $H$ + unlabeled data distribution $D$ \xrightarrow{} disagreement metric space $(\mathcal{H}, \rho_D)$
  \[
  \rho_D(h, h') = \Pr_{x \sim D}[h(x) \neq h'(x)]
  \]

- Doubling dimension is $d$ if every ball of radius $r$ can be covered by $2^d$ balls of radius $r/2$ (and no fewer).

- (Non-private) sample complexity bound due to Bshouty et al (2009) for noiseless setting:
  \[
  C \cdot \frac{1}{\epsilon} \left( d + \log \frac{1}{\delta} \right)
  \]
Divergence $\kappa(U,D)$

$$\kappa(U,D) = \inf \left\{ k > 0 : \Pr_{x \sim D}[x \in A] \leq k \cdot \Pr_{x \sim U}[x \in A] \quad \forall \text{measurable } A \right\}$$

(Quantifies absolute continuity of $D$ w.r.t. $U$.)