Three Aspects of Gödel’s Program: Supercompactness, Forcing axioms, Ω-logic

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Wien
Corollary (Todorčević, 1989)

Assume $P \subseteq \mathbb{R}^2 \setminus \Delta$ is an open symmetric set. Then exactly one of the following holds:

1. There is a closed uncountable set $C$ such that $C^2 \subseteq P$,
2. $\mathbb{R} = \bigcup_{n \in \mathbb{N}} C_n$ where each $C_n$ is a closed set and $C_n^2 \cap P = \emptyset$. 

$\Delta = \{ (x, x) : x \in \mathbb{R} \}$ is the diagonal,

Symmetric sets which do not intersect the diagonal are determined by their intersection with the half plane $H = \{ (x, y) \in \mathbb{R}^2 : x > y \}$.
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\( P \) is symmetric if \( (a, b) \in P \iff (b, a) \in P \),

\( \Sigma_1^1 \)-statement OCA*
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Open partitions of $H$. I
Open partitions of $\mathcal{H}$. II
Open partitions of $H$. III

A trivial example of case 1
Open partitions of $H$. IV

An example of case 2
Open partitions of $H. V$

A non trivial example example of case 1
Theorem (Todorčević)

Assume the proper forcing axiom PFA. Then for every $X \subseteq \mathbb{R}$ and every open and symmetric $P \subseteq \mathbb{R}^2$ exactly one of the following holds:

1. There is a closed set $C$ such that $C^2 \subseteq P$ and $C \cap X$ is uncountable,
2. There is a countable family of closed sets $C_n$ such that $X \subseteq \bigcup_{n \in \mathbb{N}} C_n$ and $C_n^2 \cap P = \emptyset$ for all $n$. 
Theorem (Shoenfield, 1961)

Assume $\phi$ is a $\Pi^1_2$-statement. If there is an uncountable transitive model $M$ of ZFC such that $M \models \phi$, then $\phi$ holds in all transitive uncountable models of ZFC.
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In general differential and algebraic geometry are usually concerned with $\Pi^1_2$-problems, the same occurs for large portions of analysis and number theory.
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In general differential and algebraic geometry are usually concerned with $\Pi^1_2$-problems, the same occurs for large portions of analysis and number theory.

On the other hand non $\Pi^1_2$-problems may show up with more frequency in general topology, functional analysis, homological algebra, category theory…
Theorem (Baumgartner, 1984)

Assume there is a model of ZFC with a supercompact cardinal. Then there is a model of ZFC + PFA.
For every open and symmetric partition $P$ of the plane $\mathbb{R}^2$ exactly one of the following holds:

1. There is an uncountable closed set $C$ such that $C^2 \subseteq P$,
2. There is a countable family of closed sets $C_n$ such that $\mathbb{R} = \bigcup_{n \in \mathbb{N}} C_n$ and $C_n^2 \cap P = \emptyset$ for all $n$.

is the $\Pi^1_2$-property

$$\forall P \phi(P) \rightarrow (\exists C \psi(P, C) \lor \exists \tilde{C} \theta(P, \tilde{C}))$$

where $\phi(P), \psi(P, C), \theta(P, \tilde{C})$ are the $\Delta^1_1$-statements:
OCA∗

For every open and symmetric partition $P$ of the plane $\mathbb{R}^2$ exactly one of the following holds:

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- $\theta(P, (C_n^2 : n \in \omega)) \equiv \ldots$
"Platonistic" proof of the Corollary

If we assume a platonistic stand-point and accept large cardinals axioms, the corollary is an immediate consequence of the three theorems since:

1. By Baumgartner's theorem:
   If there is a supercompact cardinal, then we can "safely" assume that there is an uncountable transitive model $M$ of PFA.

2. By Todorcevic's theorem:
   If PFA holds in $M$, then OCA$^*$ holds in $M$.

3. By Shoenfield's absoluteness:
   If OCA$^*$ holds in some transitive uncountable model $M$ of ZFC, then it holds in all uncountable transitive models $M$ of ZFC.

Thus OCA$^*$ is true.

A posteriori an "ordinary" proof of OCA$^*$ has been found.
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1. Forcing axioms.

2. $\Omega$-logic and absoluteness

3. Large cardinals and forcing axioms
Forcing axioms solve problems!
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Take a mathematical problem which is likely to be independent of ZFC, then there are great hopes that PFA will decide it.
Some examples from cardinal arithmetic:

The continuum hypothesis CH:
\[2^{\aleph_0} = \aleph_1.\]

Theorem (Todorˇ cevi´ c-Veliˇ ckovi´ c (1992), many others and many proofs afterwards)
PFA \[\rightarrow\] \[2^{\aleph_0} = \aleph_2.\]
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Theorem (V. (2006))

PFA → SCH
Some examples from general topology:

Souslin's Hypothesis SH:
There are no Souslin lines.

Theorem (Solovay-Tennenbaum (1971))
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Is there a regular Hausdorff space which is hereditarily separable but not hereditarily Lindelöf?
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Assume PFA. Then the answer is no.
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**Theorem (Moore (2006))**

Yes, there is.
The five element basis for the uncountable linear orders:

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Assume PFA. Then there are five uncountable linear orders such that any other uncountable linear order contains an isomorphic copy of one of these five.
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Examples from functional analysis and algebra:

- Whitehead's problem: Is every Whitehead group free?

  Theorem (Shelah (1974))
  \[ \text{Assume PFA (MA suffices). Then there is a Whitehead group which is not free.} \]

- Is every automorphism of the Calkin algebra an inner automorphism?

  Theorem (Farah, 2011, culminating researches by himself, Shelah, Velić and many others)
  \[ \text{Assume PFA. Then all automorphisms of the Calkin algebra are inner.} \]
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**Forcing** is an algorithmic procedure which takes as inputs

- a model $V$ of ZFC
- a boolean algebra $\mathbb{B} \in V$. 

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Forcing is an algorithmic procedure which takes as inputs

- a model $V$ of ZFC
- a boolean algebra $B \in V$.

From these inputs the forcing method produce a new model $V^B$ of ZFC.

Truth in $V^B$ is "computable" and depends from the combinatorial properties of $B$ and from the first order theory of $V$. 
$\Omega$-Logic.

$\Omega$-logic is devised in order to make set theory resilient to the forcing method.

Definition

$V^{|\omega} = \Omega \phi$ iff $V^B | \omega = \phi$ for all complete Boolean algebras $B \in V$.

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Definition

\[ \forall V |\models \phi \iff \forall V^B \models \phi \text{ for all complete Boolean algebras } B \in V. \]
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**Theorem (Woodin, late eighties (in print 1999))**

Assume $V$ is a transitive model of $\text{ZFC}^*$. Then for all complete Boolean algebras $\mathbb{B} \in V$ and all statements $\phi$:

$$V \models _{\Omega} \phi \iff V^\mathbb{B} \models ("V \models _{\Omega} \phi")$$

If one is eager to accept large cardinal axioms as true, $\Omega$-truth is absolute with respect to the forcing method.
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Assume $\phi$ is a $\Pi^1_n$-property. Then $\text{ZFC}^* \models \Omega \phi$ or $\text{ZFC}^* \models \Omega \neg \phi$.

More generally:

Theorem (Woodin, unpublished)

Assume $\phi$ is a mathematical statement such that $\text{ZFC} \vdash \"\phi$ is expressible as a $\Delta^2_1$-property.\" Then $\text{ZFC}^* \models \Omega \phi$ or $\text{ZFC}^* \models \Omega \neg \phi$.

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$\text{ZFC}^* \models \Omega \"L(P_{\omega1\text{Ord}}) \models \phi\"$ or $\text{ZFC}^* \models \Omega \"L(P_{\omega1\text{Ord}}) \models \neg \phi\"$.

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Assume $\phi$ is a $\Pi^1_n$-property. Then $\text{ZFC}^* \models \Omega \phi$ or $\text{ZFC}^* \models \Omega \neg \phi$.

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Why large cardinals settle the theory of $L(\mathbb{R})$?

Theorem (Martin, Steel (1988))

Assume $\text{ZFC}^*$. Then the axiom of determinacy $\text{AD}$ holds in $L(\mathbb{R})$.

It is well known that any mathematical problem which is expressible as a $\Pi_1^n$-property has very high chances to be settled by $\text{AD}$.

For example:

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Assume $\text{AD}$. Then every set of reals has the Baire property and is either countable or contains a closed uncountable set.
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3. Diamond or CH?
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**Problem**

Assume PFA holds in a transitive model $V$. Is there a transitive inner model of $V$ with a supercompact cardinal?
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There is a "canonical" inner model for a supercompact cardinal if and only if such a canonical model can be built assuming PFA.
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If it were possible to show that inner models for large cardinals are simply definable assuming strong forcing axioms this would give more ground to accept them as a reasonable strengthening of the notion of large cardinal or even as "generic large cardinals axioms".
THANK YOU FOR YOUR ATTENTION