PAST, PRESENT, AND FUTURE DIRECTIONS IN THE FOUNDATIONS OF MATHEMATICS

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AGENDA

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5. Fundamental Foundational Moves: consistency, completeness, incompleteness.

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THE FOUNDATIONAL LIFE

Kurt Gödel is the great practitioner of the Foundational Life who has had such a profound influence on me and so many others.

In the Foundational Life, there are two principal goals.

i. The founding of new systematic disciplines that have the same general features as the great systematic disciplines that have emerged over the centuries - such as mathematics, physics, statistics, computing science, electrical engineering, etc.

ii. The redevelopment, reorganization, and imaginative exposition of existing systematic disciplines in the direction of creating deeper relevance to the original purposes for which they were founded.

In the Foundational Life, philosophy is commonly used as a method for choosing and analyzing fundamental concepts, and mathematics is commonly used for rigorous development. The mathematics informs the philosophy and the philosophy informs the mathematics.

In the Foundational Life, there is a delicate balance between the philosophical and the mathematical, both subservient to the principal focus on i,ii above.

FOUNDATIONAL LIFE

In the Foundational Life, no premium is paid, per se, to careful philosophical statements and arguments, and no premium is paid, per se, to complicated or deep mathematical developments. However, careful philosophy and deep mathematics are used if they facilitate the main focus on i,ii - and they often do.

In the Foundational Life, there is constant assessment of the prospects for developing new systematic disciplines, or redevelopment of existing systematic disciplines. Generally, this requires that there be a wide range of deep phenomena available for analysis.

The Foundational Life as I practice it, is both highly mathematical and highly philosophical. But it differs profoundly from the Mathematical Life and the Philosophical Life in various ways.

PHILOSOPHICAL LIFE

In the Philosophical Life, there is a focus on careful analysis of <u>apparently</u> fundamental concepts, and making careful arguments that can be defended against attacks, as well as attacking other's careful arguments.

In the Philosophical Life, the choice of these <u>apparently</u> fundamental concepts is not generally measured in terms of their appropriateness for the creation of new systematic disciplines that have the same general features as the great systematic disciplines that have emerged over the centuries - mathematics, physics, statistics, computing science, etc.

The clearest exception to the normal modus operandi in the Philosophical Life is the emergence of Foundations of Mathematics. This involved the analysis of such truly fundamental concepts as rigorous proof, natural number, deterministic algorithm, etc., and involved such great practitioners of the Philosophical Life as Aristotle, Frege, Russell, and others.

MATHEMATICAL LIFE
SCIENTIFIC LIFE

In the Mathematical Life, there is a focus on the rigorous development of detailed information within well defined frameworks that have emerged for a variety of purposes, often from outside mathematics.

These developments take on a life of their own, without generally being evaluated in light of the original purposes that generated them. Instead, developments are normally evaluated in terms of beauty, originality, and complexity.

The Scientific Life, as normally practiced in academia, generally comes in at least two very different forms: Experimental Scientific Life and Theoretical Scientific Life. Of course, the most profound figures were able to combine these two Lives in path breaking ways.

My understanding of science has not been sufficiently deep for my Foundational Life to interact with the Scientific Life. I have found too many irresistible opportunities present in the Foundational Life related to Mathematics - and, to a lesser, limited extent, computer science, education, and music.

AN AMBITION

But I do have the ambition of applying the lessons that we can learn from the great and profound successes in Foundations of Mathematics, to the future Foundations of Science.

Bringing the Foundations of Physical Science up to anything like the level of where Gödel saw the Foundations of Mathematics as a student in Vienna, is going to require a profound rethinking of the most elemental aspects of both mathematics and science in ways that we can only begin to imagine.

This is not meant as an insult to physical scientists. Foundations of Physical Science appears to be transcendentally more difficult than Foundations of Mathematics - which has proved difficult enough.

It took till the early part of the 20th century for the Foundations of Mathematics to become the great intellectual structure that it is - with mathematical logic being spun off as a substantial area of mathematics, now with significant applications to other areas of mathematics.

Yet mathematics has had a very long development, starting in antiquity. There was already a vast development of mathematics well before Foundations of Mathematics took hold in anything like its present form.

FOUNDATIONS OF MATHEMATICS
FOUNDATIONS OF PHYSICAL SCIENCE

In the Foundational Life, one should expect to experience very long gestation periods, with a great deal of trial and error. It seems to require multidisciplinary people who think very differently than the mainstream, with great imagination, power, rigor, and inspiration.

Among all governmental, academic, and philanthropic institutions worldwide, I know of only one that might be persuaded to invest carefully and wisely towards

bringing the Foundations of Physical Science
up to anything resembling the early levels
of the Foundations of Mathematics

which will require deep, organized, and intense collaborative efforts involving the right kinds of open minded mathematicians, logicians, philosophers, computer scientists, and physical scientists.

And that one institution that may be persuadable is the John Templeton Foundation.

FOUNDATIONS OF APPLIED MATHEMATICS

I have heard that the prevailing attitude among physical scientists is

shut up and calculate!

The prevailing attitude among mathematicians is, correspondingly,

shut up and prove!

In the case of the Foundations of Mathematics, this wide range of deep phenomena consists of mathematical practice - both pure and applied. I believe that Foundations of Applied Mathematics is grossly underdeveloped, and will play a major role in the Foundational Life of the future.

PROFOUND UNEASINESS

A profound uneasiness in mathematics, say around 1800, set the stage for the emergence of the Foundations of Mathematics.

There is a profound uneasiness in physical science now, which is fully recognized by the "it doesn't seem to make any sense; what does it all mean?" crowd, if not recognized by the "shut up and calculate!" crowd.

I expect that in the future, we will be able to make fruitful connections between these two cases of profound uneasiness, which are sufficient to let Foundations of Mathematics lead the way by example to breakthrough developments in the Foundations of Physical Science.

To set the stage for the emergence of Foundations of Mathematics, consider what mathematics looked like around 1800. For example, here is a quote taken from M. Kline, Mathematical Thought from Ancient to Modern Times, p. 947:

PROFOUND UNEASINESS

"By about 1800 the mathematicians began to be concerned about the looseness in the concepts and proofs of the vast branches of analysis. The very concept of a function was not clear; the use of series without regard to convergence and divergence had produced paradoxes and disagreements; the controversy about the representations of functions by trigonometric series had introduced further confusion; and, of course, the fundamental notions of derivative and integral had never been properly defined. All these difficulties finally brought on dissatisfaction with the logical status of analysis."

This profound uneasiness led directly to what is called "the installation of rigor in mathematics".

FIRST MAIN STEP
EPSILON DELTA

The first main step was to make a series of fundamental definitions, based on the still unanalyzed number systems, taking their basic properties for granted.

There was gradual emergence and acceptance during the 19th century of the fundamental definitions we use today, such as

i. Limit of a sequence of reals.
ii. Sum of an infinite series of reals.
iii. Limit of a real function at a point.
iv. Continuity of real functions.
v. Derivative of a real function at a point.
vi. Definite integral of a real function over an interval.

through the so called epsilon/delta methodology.

NUMBER SYSTEMS

Eventually, there was full realization that some of these entities may not exist, and that tacit assumptions of existence were responsible for a number of paradoxes and confusions. For instance

$$1 + -1 + 1 + -1 \ldots$$

does not exist, avoiding paradoxes emanating from the rearrangement of terms. Or that if something like the above is to exist, then it must be given an unambiguous definition, which must be consistently adhered to.

A new standard of proof emerged, in which such definitions were fully incorporated.

However, it was not until the late 19th century that the number systems, and the function concept, were appropriately rigorized. For instance, Weierstrass is credited for first realizing that to establish the properties of continuous functions, one needs a rigorous treatment of the real number system.

REAL NUMBER SYSTEM
MULTIPLE DEFINITIONS

Two approaches emerged in the late 19th century for rigorously developing the real number system. They both have their advantages and disadvantages. One is the Dedekind cut definition, and the other is the Cauchy sequence approach. The Dedekind cut definition involves sets of rational numbers, and the Cauchy sequence definition involves sequences of rational numbers.

In the Dedekind cut approach, real numbers are literally certain sets of rationals (left cuts). But the treatment of multiplication is unnatural.

In the Cauchy sequence approach, real numbers are literally certain sets of sequences of rationals - the equivalence classes of Cauchy sequences of rationals under an obvious equivalence relation. Addition and multiplication are both natural.

A confusing aspect of the foundations of mathematics is the emergence of multiple definitions for the same concept. Obviously these multiple definitions cannot be taken literally.

INFINITE UPPER SHIFT KERNEL THEOREM

Good ways of looking at the phenomena of multiple definitions had to wait until the 20th century, where appropriate notions of mathematical structure and isomorphism were developed.

For example, the real number system is unique up to isomorphism as a complete ordered field.

However, this so called synthetic approach raises additional issues: where do these structures come from, and where do the isomorphisms between them come from?

In fact, the "right" way to set up the synthetic approach, systematically, is still a topic of controversy and research to this day.

Coming back to the development of the real number system through Dedekind cuts and Cauchy sequences, notice how it relies on the "more elementary" rational number system.

RATIONALS AND INTEGERS

Rationals are normally defined as either ordered pairs of integers in reduced form, or equivalence classes of ordered pairs of integers. Synthetically, rationals form the unique ordered field with no proper subfield, under isomorphism.

The integers, in turn, are defined as either ordered pairs of natural numbers in reduced form, or signed natural numbers, or equivalence classes of pairs of natural numbers.

Synthetically, the integers form the unique commutative ordered ring with no proper subring, under isomorphism.

The definition of the natural numbers creates additional issues. There is a synthetic definition due to Peano. But, ultimately, synthetic definitions need a foundation.

Although at this stage, the foundations of mathematics was greatly clarified from the chaos and confusion of 1800, a major new simplifying idea was needed in order to create the sought after ultimate foundations.

GRAND UNIFICATION

The grand unifying idea for foundations of mathematics has proved to be the set concept.

The truly unifying set concept is not the ordinary one - sets of "atomic" objects. It is a more sophisticated one where the elements of sets are not restricted, and may be themselves sets.

Set theory was intensively developed as a branch of mathematics by Georg Cantor starting in the late 19th century.

In pure set theory, all objects are sets. The only concept is that of membership.

Equality can be treated in one of two ways, that are essentially equivalent. i) below has been adopted.

i. We can take equality as a primitive concept, as we can do in all contexts. We assert that if two sets have the same elements, then they are equal.

ii. We can avoid using equality at all. We assert that if two sets have the same elements, then they are elements of the same sets.

PURE SET THEORY
IMMUTABLE OBJECTS

Pure set theory constitutes a very bold grand unification. Yet it works very well, and has stood the test of time. It is based on objects that are unchanging, with completely objective properties. No time, and no observer!

This immutability is very striking. Consider the familiar case of a moving point, which might model a thrown projectile. Instead of working directly with a single point that is changing its position (a changing property of the point), we instead work with a single object that is unchanging. This single object is much more complicated than a mere point. This single object is of course a function from an interval in the reals (the relevant time interval) into reals (the position).

This move from changing points to fixed immutable objects appears as a crucial step towards rigor. The resulting function is analyzed in terms of limits, derivatives, integrals, local/absolute extrema, etc.

EXPLODING UNIVERSE

Yet this move, and perhaps other related moves, may represent the basis for a disconnect between physical science and mathematics. Perhaps, now that we have such deep understanding of the foundations of mathematics, we should experiment with enriching set theory thru the direct incorporation of changing objects. Even more radical would be to introduce the observer into set theory.

There are indications of what can be gained by incorporating even a very limited form of changing objects into mathematics. I considered what I called the exploding set theoretic universe. There are two set theoretic universes. One now, and one later, which is formed upon an explosion. Principles relating the two universes are proposed. This situation supports the construction of models of set theory in which so called "large cardinal hypotheses" hold.

I expect that "the evolving universe" - i.e., "the evolving set theoretic universe", or "the evolving mathematical universe", will become a major topic in future Foundational Life.

ASSERTIONS AND PROOFS

With the establishment of the interpretation of mathematics in set theory, the current foundations of mathematics took shape. However, there is a crucial missing element. This is logical structure.

The appropriate logical structure for Foundations of Mathematics is given by what is now called first order predicate calculus (with equality). This calculus is generally credited to Gottlob Frege from the late 19th century.

Logical structure, at least ideally, applies to all deductive reasoning, regardless of whether it is confined to mathematics.

However, in practice, there is a utter lack of substantial examples of deductive reasoning with anything like the depth and complexity of deductive reasoning in mathematics.

Instead, outside mathematics, we almost entirely rely on instincts and common sense. This is true, generally speaking, even in the realm of science.

LOGICAL STRUCTURE

Consider all of the reasoning that goes into the design of delicate and complex experiments, which are to confirm or refute theories. There is a vast array of hidden assumptions, most of which resist clear formulation.

I expect that the logical analysis of scientific experimentation will become a major component in the Foundational Life of the future.

Returning to logical structure within mathematics, we first need the notion of an assertion. For this, we use

variables ranging over sets
membership and equality ($\in$,=)
connectives (not, and, or, if then, if and only if)
quantifiers (for all, there exists)

We then have the so called axioms and rules of inference for logic. These are to apply to any situation - they don't depend on what sets are, or what membership means. They do depend on the meaning of =.

PROOF ASSISTANTS

Firstly, an elaborate system of abbreviations and conventions have developed in order to support the construction of actual mathematical assertions.

Secondly, mathematicians generally require a massive infusion of additional axioms and rules of inference of logic, which are not theoretically new (by Gödel's Completeness Theorem no appropriate new ones exist), but which are needed in the practical sense.

Thirdly, it is just too burdensome for a human to take care of every detail in a non straightforward argument.

All of this leads to the obvious question of whether formally correct proofs actually have been constructed - or even can be actually constructed - for substantial mathematical theorems.

The answer is yes, but with the help of a computer. The biggest inventory of actual formally correct proofs emanates from the proof assistant called Mizar. There are rivals; e.g., Isabelle, Coq, HOL, etc.

Systems like Mizar keep track of and help supply details. However, at present, they are very limited and primitive.

UNDERSTANDING TRIVIALITIES

What is badly needed is a better understanding of "trivial inferences", where the computer supplies the trivial inferences, and the human supplies the nontrivial inferences. One also needs a much richer supply of humanly created algorithms for fragments of mathematics that can be applied automatically and effectively by the computer.

I expect that the development of algorithms for fragments of mathematics, and an understanding of trivialities, will be a major part of the future of the foundations of mathematics, where the creation of formally correct proofs will be greatly facilitated and expanded.

I further expect that this expansion of the inventory of formally correct proofs will lead to a new level of understanding of the structure of actual mathematical proofs.

So now we have that gold standard of mathematical proof - the ZFC axiom system. But experience has shown that ZFC is vast overkill for the vast preponderance of mathematics.

This led me to the development of so called Reverse Mathematics, which robustly classifies mathematical theorems according to the logical principles needed for their proof.

STRICT REVERSE MATHEMATICS

This classification is supported by the fact that so many mathematical theorems are demonstrably equivalent to logical principles far weaker than ZFC. In Reverse Mathematics, logical principles are proved from mathematical theorems - hence the name Reverse Mathematics.

I introduced the base theory, RCA_0, for Reverse Mathematics, over which the "reversals" are made. I set up the field in the late 1960s to mid 1970s based on RCA_0.

Ideally there should be no base theory - or any base theory should consist solely of mathematical assertions that are explicitly essential in all of mathematics. E.g., the discrete order ring axioms for the integers.

I already envisioned, and wrote about this kind of Strict Reverse Mathematics, before I set up Reverse Mathematics with RCA_0. I chose the present setup because of its special clarity and problem generating power. In fact, it is now clear that Strict Reverse Mathematics would have been premature at that time.

I expect that the further development of Strict Reverse Mathematics that I have begun recently will be a major part of the future of the Foundations of Mathematics.

FUNDAMENTAL FOUNDATIONAL MOVES: consistency, completeness, incompleteness

The issues of consistency, completeness, and incompleteness have framed the major developments in the Foundations of Mathematics since the time of Kurt Gödel.

A formal system is consistent if and only if it does not prove a contradiction.

The idea is that if a system is inconsistent, then it is worthless because it can prove all statements - and therefore makes no contribution to separating the true from the false.

The idea has been floated that if we remove the logical inference that

from a contradiction, we can derive any statement

called explosion, then an inconsistent system might still have some value in that it may not prove all statements.

INCONSISTENT SYSTEMS USEFUL?

At present, this intriguing proposal lacks sufficient justification. For instance, in http://en.wikipedia.org/wiki/Paraconsistent_logic#Tradeoff it is reported that the rejection of explosion entails the rejection of at least one of three principles, each one of which is definitely used in actual mathematics.

A counter might be to argue that mathematics can be appropriately developed without, for example: from A and $\neg A \vee B$, derive B. However, this has not been established.

There is a much clearer way in which an inconsistent system can be of value. That is, where all proofs of a contradiction are of ridiculously enormous size. Unfortunately, this does not represent any kind of solution to paradoxes such as the Russell paradox or the Burali-Forti paradox, since they involve only proofs of small size.

GÖDEL'S SECOND INCOMPLETENESS THEOREM

Gödel's second incompleteness theorem, formulated in modern terms, establishes that

Con(ZFC) cannot be proved in ZFC,
unless ZFC is in fact inconsistent

where Con(T) is "T is consistent". One version of Hilbert's program is to establish Con(ZFC) within PA = Peano Arithmetic, or even a weak fragment T of PA, where Hilbert regarded Con(T) as indisputable. But then from Gödel's second incompleteness theorem, we have

Con(ZFC) cannot be proved in T.

We now review four great Completeness Theorems, due to Gödel, Presburger, and Tarski.

FOUR COMPLETENESS THEOREMS

COMPLETENESS THEOREM 1. (Gödel 1929). A sentence in predicate calculus is true in all structures if and only if it is provable from the usual axioms and rules of inference of logic.

The above theorem establishes that the usual axioms and rules of inference of logic = LOGIC, are not subject to any expansion that is compatible with its intended purpose.

COMPLETENESS THEOREM 2. (Presburger 1929). A sentence about the ordered group of integers is true if and only if it is provable from the axioms for discrete ordered groups, and the quotient remainder axioms, combined with LOGIC.

COMPLETENESS THEOREM 3. (Tarski 1951). A sentence about the ordered field of real numbers is true if and only if it is provable from the axioms for ordered real closed fields combined with LOGIC.

COMPLETENESS THEOREM 4. (Tarski 1951). A sentence about the field of complex numbers is true if and only if it is provable from the axioms for algebraically closed fields of characteristic zero combined with LOGIC.

GÖDEL'S FIRST INCOMPLETENESS THEOREM

We now turn to Gödel's first incompleteness theorem.

INCOMPLETENESS THEOREM 1. (Gödel 31). Let T be a consistent extension of a weak fragment of Peano Arithmetic known as Raphael Robinson's Q. Assume that the axioms of T consist of finitely many axioms and axiom schemes. There is a sentence that is neither provable nor refutable in T.

We can obtain the following as a Corollary.

INCOMPLETENESS THEOREM 2. (Gödel 31). There is no finite set of axioms and axiom schemes which, when combined with the usual axioms and rules of inference of logic, proves exactly the true sentences about the ring of integers.

INCOMPLETENESS THEOREM 3. (Julia Robinson 49). There is no finite set of axioms and axiom schemes which, when combined with the usual axioms and rules of inference of logic, proves exactly the true sentences about the field of rationals.

FIRST MATHEMATICALLY NATURAL INCOMPLETENESS

The first example of a mathematically natural assertion that is neither provable nor refutable in ZFC is as follows.

INCOMPLETENESS THEOREM 4. (Gödel 31, Cohen 63,64). The sentence "every infinite set of reals is in one-one correspondence with the integers or the real numbers" is neither provable nor refutable in ZFC, unless ZFC is inconsistent.

CONSISTENCY, AND THE INCORPORATION OF NEW NOTIONS

I am expecting that some natural condition on proofs in ZFC will be discovered, where

i. It is shown that all proofs before 2011 in the published mathematical literature are proofs in ZFC obeying the natural condition.
ii. It is provable in ZFC, or even in PA, that there is no proof of a contradiction in ZFC obeying the natural condition.

I expect that the proof in ZFC or in PA in ii) would not obey the natural condition. In fact, this is probably a general result assuming that the natural condition itself obeys some natural conditions.

Many years ago, I discovered a finite form of Gödel's second incompleteness theorem, which has subsequently been refined by Pavel Pudlak. It asserts, roughly speaking, that

Any proof in PA that there is
no short inconsistency proof in ZFC,
must be long,
unless ZFC in fact has a short inconsistency proof.

CONCEPT CALCULUS

However, the results thus far do not bear directly on actual mathematical practice, since the results are asymptotic. It would appear that when the asymptotics are removed, the numbers reflect large overhead, weakening the practical import of the results. There is also the issue of realistic proof systems that adequately reflect mathematical practice. Finally, there are issues related to the P = NP problem that will limit the strength of the results, at least until (finite forms of) P = NP is resolved.

I have been developing Concept Calculus, in which basic common sense concepts from outside science are logically analyzed and given plausible axiomatizations. I then show that these resulting axiomatizations are mutually interpretable with various set theories, including ZFC and ZFC extended by the so called large cardinal hypotheses.

As a Corollary, this provides consistency proofs of ZFC using the consistency of such axiomatizations. If these axiomatizations are suitably augmented with nonproblematic principles involving the natural numbers, then these axiomatizations become sufficient to state and prove Con (ZFC).

COMPLETENESS, AND DELICATE CHOICE OF FRAGMENTS

I expect Concept Calculus to become a major part of the Foundational Life, forging unexpected new links between mathematics and philosophy.

The complete axiomatizations of the ordered group of integers, the ordered real closed field of reals, and the field of complex numbers of characteristic zero, were proved by the method of quantifier elimination.

This is a fundamental technique used by model theorists in many contexts. Furthermore, they do not emphasize completeness, but rather the properties of the definable sets in such structures.

In particular, they have formulated a fundamental property of linearly ordered structures known as 0-minimality. This asserts that every definable subset of the domain is a finite union of intervals with endpoints in the domain (infinite endpoints are allowed).

A surprising array of properties involving the higher dimensional definable sets follow by very interesting arguments just from 0-minimality.

0-MINIMALITY

Yet completeness is not a consequence of 0-minimality even in the fundamental case of the ordered field of real numbers with exponentiation added - which is known to be 0-minimal (Wilkie 1996). Unfortunately, even deciding the equalities between exponential constants constitutes hopelessly difficult number theory at the present time.

However, there is a famous number theoretic conjecture called Schanuel's Conjecture. It is way out of reach at the present time, and it is very powerful.

COMPLETENESS 5. (MacIntyre, Wilkie 1996). Every sentence about the ordered field of reals together with the exponential function, is provable or refutable in ZFC + Schanuel's Conjecture. In fact, ZFC can be replaced by an explicitly given natural set of axioms.

I expect that a number of developments in completeness will play a substantial role in the Foundations of Mathematics in the future. These include the following.

0-MINIMALITY

i. An understanding of when the expansion of the ordered field of real numbers by a real power series is 0-minimal, in terms of basic properties of the power series.

ii. An understanding of the scope of 0-minimal expansions of the ordered field of reals, including the issue of the possible growth rates at infinity.

iii. A finite form for 0-minimality, revealing its underlying finite combinatorial content.

iv. The systematic development of completeness theorems throughout mathematics, without requiring 0-minimlaity. Various fruitful weakenings of 0-minimality have been fruitfully explored (C. Miller). These still involve considering all definable subsets of the domain. I expect various imaginative restrictions on the sentences considered in various context to emerge, that support new kinds of completeness theorems.

INCOMPLETENESS, AND CONCRETENESS, SIMPLICITY, NATURALNESS

Concrete Mathematical Incompleteness is discussed at great length in the Introduction to my book Boolean Relation Theory and Incompleteness, available on my website.

Is there a mathematically natural concrete mathematical statement which cannot be proved or refuted in ZFC?

This is the basic issue in Concrete Mathematical Incompleteness that is so crucial for the future of the Incompleteness Phenomenon, and for Foundations of Mathematics generally.

Here is the state of the art. Below, all intervals are intervals in the rationals.

MAXIMAL CLIQUE EMBEDDING

In work generously supported by the John Templeton Foundation:

MAXIMAL CLIQUE EMBEDDING THEOREM. Every order invariant graph on $[0,1]^k$ has a maximal clique with a nontrivial embedding whose range and fixed points each form an interval.

EXPLICIT MAXIMAL CLIQUE EMBEDDING THEOREM. Every order invariant graph on $([-1,1]\backslash\{0\})^k$ has a maximal clique with the embedding $\min(q,q/2)$, $q \in [-1,1]\backslash\{0\}$.

These are neither provable nor refutable in ZFC. They are equivalent to the consistency of Mahlo cardinals of finite order.

I expect that the examples will permeate the whole of mathematics, and some of them will be explicitly finite.

I expect there to emerge a theory of mathematical naturalness, spearheaded by the construction of languages based on exceedingly common mathematical notions. Mathematical Naturalness will then be successfully modeled in terms of the simplicity of assertions within such languages.