Empirical portfolio selections

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fixed portfolio selection (single period)
Portfolio selections

- fixed portfolio selection (single period)
- constantly rebalanced portfolio
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- general rebalancing
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- rebalancing
- multi-asset, multi-period
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- fixed portfolio selection (single period)
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- rebalancing
- multi-asset, multi-period
- empirical (nonparametric statistics, machine learning)
investment in the stock market $d$ assets
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$d$ assets
$S_n^{(j)}$ price of asset $j$ at the end of trading period (day) $n$,
$j = 1, \ldots, d$
Growth rate

investment in the stock market

\(d\) assets

\(S_n^{(j)}\) price of asset \(j\) at the end of trading period (day) \(n\),

\(j = 1, \ldots, d\)

\(S_0^{(j)} = 1\)
growth rate

investment in the stock market

$d$ assets

$S_n^{(j)}$ price of asset $j$ at the end of trading period (day) $n$,

$j = 1, \ldots, d$

$S_0^{(j)} = 1$

\[
S_n^{(j)} = e^{nW_n^{(j)}} \approx e^{nW^{(j)}}
\]
Growth rate

investment in the stock market
$d$ assets
$S^{(j)}_n$ price of asset $j$ at the end of trading period (day) $n$,
$j = 1, \ldots, d$
$S^{(j)}_0 = 1$

$$S^{(j)}_n = e^{nW^{(j)}_n} \approx e^{nW^{(j)}}$$

asymptotic growth rate

$$W^{(j)} = \lim_{n \to \infty} W^{(j)}_n = \lim_{n \to \infty} \frac{1}{n} \ln S^{(j)}_n$$
the aim is to achieve $\max_j W(j)$
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static portfolio selection
the aim is to achieve $\max_j W^{(j)}$

static portfolio selection

a portfolio vector $\mathbf{b} = (b^{(1)}, \ldots b^{(d)})$

$b^{(j)} \geq 0$, $\sum_j b^{(j)} = 1$
the aim is to achieve $\max_j W^{(j)}$
static portfolio selection
a portfolio vector $\mathbf{b} = (b^{(1)}, \ldots b^{(d)})$
$b^{(j)} \geq 0$, $\sum_j b^{(j)} = 1$

$$S_n = \sum_j b^{(j)} S^{(j)}_n$$

we can do much better, applying dynamic portfolio selection
the aim is to achieve \( \max_j W^{(j)} \)

static portfolio selection

a portfolio vector \( \mathbf{b} = (b^{(1)}, \ldots b^{(d)}) \)

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\[
S_n = \sum_j b^{(j)} S_n^{(j)}
\]

\[
\max_j b^{(j)} S_n^{(j)} \leq S_n \leq d \max_j b^{(j)} S_n^{(j)}
\]
the aim is to achieve $\text{max}_j W(j)$
static portfolio selection
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$b^{(j)} \geq 0$, $\sum_j b^{(j)} = 1$

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$$\lim_{n \to \infty} \frac{1}{n} \ln S_n = \lim_{n \to \infty} \max_j \frac{1}{n} \ln S^{(j)}_n = \text{max}_j W^{(j)}$$
the aim is to achieve $\max_j W(j)$

static portfolio selection

a portfolio vector $b = (b^{(1)}, \ldots b^{(d)})$

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we can do much better, applying dynamic portfolio selection
$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$
Dynamic portfolio selection: multi-period investment

\[ x_i^{(j)} = \frac{S_i^{(j)}}{S_i^{(j)} - S_{i-1}} \]

\( x_i = (x_i^{(1)}, \ldots, x_i^{(d)}) \) the return vector on day \( i \)
Dynamic portfolio selection: multi-period investment

\[ x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}} \]

\[ x_i = (x_i^{(1)}, \ldots, x_i^{(d)}) \] the return vector on day \( i \)

\( x_i^{(j)} \) is the factor by which capital invested in stock \( j \) grows during the market period \( i \)
Dynamic portfolio selection: multi-period investment

\[ x_i(j) = \frac{S_i(j)}{S_{i-1}} \]

\( x_i = (x_i^{(1)}, \ldots x_i^{(d)}) \) the return vector on day \( i \)

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multi-period investment
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multi-period investment

Constantly Re-balanced Portfolio (CRP)

a portfolio vector \( b = (b^{(1)}, \ldots b^{(d)}) \)
Dynamic portfolio selection: multi-period investment

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\[ \mathbf{b} = (b^{(1)}, \ldots b^{(d)}) \]

\( b^{(j)} \) gives the proportion of the investor’s capital invested in stock \( j \)
Dynamic portfolio selection: multi-period investment

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multi-period investment

Constantly Re-balanced Portfolio (CRP)

a portfolio vector \( b = (b^{(1)}, \ldots b^{(d)}) \)
\( b^{(j)} \) gives the proportion of the investor’s capital invested in stock \( j \)
\( b \) is the constant portfolio vector for each trading day
for the first day $S_0$ denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x^{(j)} = S_0 \langle b, x_1 \rangle$$
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$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle b, x_1 \rangle$$

for the second day, $S_1$ new initial capital

$$S_2 = S_1 \cdot \langle b, x_2 \rangle = S_0 \cdot \langle b, x_1 \rangle \cdot \langle b, x_2 \rangle.$$
for the first day $S_0$ denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b(j)x_1^{(j)} = S_0 \langle b, x_1 \rangle$$

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for the $n$th day:

$$S_n = S_{n-1} \langle b, x_n \rangle = S_0 \prod_{i=1}^{n} \langle b, x_i \rangle$$

with the average growth rate $W_n(b) = \frac{1}{n} \sum_{i=1}^{n} \ln \langle b, x_i \rangle$. 

Györfi

Empirical portfolio selections
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for the $n$th day:

$$S_n = S_{n-1} \langle b, x_n \rangle = S_0 \prod_{i=1}^{n} \langle b, x_i \rangle = S_0 e^{nW_n(b)}$$
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with the average growth rate

$$W_n(b) = \frac{1}{n} \sum_{i=1}^{n} \ln \langle b, x_i \rangle.$$
Special market process: $X_1, X_2, \ldots$ is independent and identically distributed (i.i.d.)
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log-optimum portfolio $b^*$

$$E\{\ln \langle b^*, X_1 \rangle \} = \max_b E\{\ln \langle b, X_1 \rangle \}$$
Special market process: $X_1, X_2, \ldots$ is independent and identically distributed (i.i.d.)

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Best Constantly Re-balanced Portfolio (BCRP)
Special market process: $X_1, X_2, \ldots$ is independent and identically distributed (i.i.d.)

log-optimum portfolio $b^*$

$$E\{\ln \langle b^*, X_1 \rangle \} = \max_b E\{\ln \langle b, X_1 \rangle \}$$

Best Constantly Re-balanced Portfolio (BCRP)

for non i.i.d. market process we can do even better
gambling, horse racing, information theory

Kelly (1956)
Latané (1959)
Breiman (1961)
Finkelstein and Whitley (1981)
Barron and Cover (1988)

”Conclusions about multiperiod investment situations are not mere variations of single-period conclusions – rather they offer reverse those earlier conclusions. This makes the subject exiting, both intellectually and in practice. Once the subtleties of multiperiod investment are understood, the reward in terms of enhanced investment performance can be substantial.”

"Conclusions about multiperiod investment situations are not mere variations of single-period conclusions – rather they offer reverse those earlier conclusions. This makes the subject exiting, both intellectually and in practice. Once the subtleties of multiperiod investment are understood, the reward in terms of enhanced investment performance can be substantial."

"Fortunately the concepts and the methods of analysis for multiperiod situation build on those of earlier chapters. Internal rate of return, present value, the comparison principle, portfolio design, and lattice and tree valuation all have natural extensions to general situations. But conclusions such as volatility is "bad" or diversification is "good" are no longer universal truths. The story is much more interesting."
Dynamic portfolio selection: general case

\[ x_i = (x_i^{(1)}, \ldots x_i^{(d)}) \] the return vector on day \( i \)
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\[ x_i = (x_i^{(1)}, \ldots, x_i^{(d)}) \] the return vector on day \( i \)
\[ b = b_1 \] is the portfolio vector for the first day
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initial capital \( S_0 \)

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S_1 = S_0 \cdot \langle b_1, x_1 \rangle
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for the second day, \( S_1 \) new initial capital, the portfolio vector
\( b_2 = b(x_1) \)

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S_2 = S_0 \cdot \langle b_1, x_1 \rangle \cdot \langle b(x_1), x_2 \rangle.
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Dynamic portfolio selection: general case

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S_1 = S_0 \cdot \langle \mathbf{b}_1, \mathbf{x}_1 \rangle
\]

for the second day, \( S_1 \) new initial capital, the portfolio vector
\( \mathbf{b}_2 = \mathbf{b}(\mathbf{x}_1) \)

\[
S_2 = S_0 \cdot \langle \mathbf{b}_1, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}(\mathbf{x}_1), \mathbf{x}_2 \rangle.
\]

nth day a portfolio strategy \( \mathbf{b}_n = \mathbf{b}(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) = \mathbf{b}(\mathbf{x}_{n-1}) \)

\[
S_n = S_0 \prod_{i=1}^{n} \langle \mathbf{b}(\mathbf{x}_{i-1}), \mathbf{x}_i \rangle
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Dynamic portfolio selection: general case

\( \mathbf{x}_i = (x^{(1)}_i, \ldots, x^{(d)}_i) \) the return vector on day \( i \)

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\[
S_n = S_0 \prod_{i=1}^{n} \langle \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_i \rangle = S_0 e^{nW_n(B)}
\]

with the average growth rate

\[
W_n(B) = \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_i \rangle.
\]
$X_1, X_2, \ldots$ drawn from the vector valued stationary and ergodic process
$X_1, X_2, \ldots$ drawn from the vector valued stationary and ergodic process

log-optimum portfolio $B^* = \{b^*(\cdot)\}$

$$\mathbb{E}\{\ln \langle b^*(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} \} = \max_{b(\cdot)} \mathbb{E}\{\ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} \}$$
Optimality

Algoet and Cover (1988): If $S^*_n = S_n(B^*)$ denotes the capital after day $n$ achieved by a log-optimum portfolio strategy $B^*$, then for any portfolio strategy $B$ with capital $S_n = S_n(B)$ and for any stationary ergodic process $\{X_n\}_{-\infty}^{\infty}$,

$$\limsup_{n \to \infty} \frac{1}{n} \ln \frac{S_n}{S^*_n} \leq 0 \quad \text{almost surely}$$
Algoet and Cover (1988): If $S_n^* = S_n(B^*)$ denotes the capital after day $n$ achieved by a log-optimum portfolio strategy $B^*$, then for any portfolio strategy $B$ with capital $S_n = S_n(B)$ and for any stationary ergodic process $\{X_n\}_{-\infty}^\infty$,

$$\limsup_{n \to \infty} \frac{1}{n} \ln \frac{S_n}{S_n^*} \leq 0 \quad \text{almost surely}$$

and

$$\lim_{n \to \infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely},$$

where

$$W^* = E \left\{ \max_{b(\cdot)} E\{ \ln \langle b(X_{-\infty}^{-1}), X_0 \rangle \mid X_{-\infty}^{-1} \} \right\}$$

is the maximal growth rate of any portfolio.
Proof

\[
\frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \langle b(X_1^{i-1}), X_i \rangle
\]
Proof

\[ \frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle b(X_{i-1}^i), X_i \right\rangle \]

\[ = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{\ln \left\langle b(X_{1}^{i-1}), X_i \right\rangle | X_{1}^{i-1}\} \]

\[ + \frac{1}{n} \sum_{i=1}^{n} \left( \ln \left\langle b(X_{1}^{i-1}), X_i \right\rangle - \mathbb{E}\{\ln \left\langle b(X_{1}^{i-1}), X_i \right\rangle | X_{1}^{i-1}\} \right) \]
Proof

\[
\frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle b(X_i^{i-1}), X_i \right\rangle
\]

= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left\{ \ln \left\langle b(X_i^{i-1}), X_i \right\rangle \mid X_i^{i-1} \right\}

+ \frac{1}{n} \sum_{i=1}^{n} \left( \ln \left\langle b(X_i^{i-1}), X_i \right\rangle - \mathbb{E}\left\{ \ln \left\langle b(X_i^{i-1}), X_i \right\rangle \mid X_i^{i-1} \right\} \right)

\text{and}

\[
\frac{1}{n} \ln S_n^* = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left\{ \ln \left\langle b^*(X_i^{i-1}), X_i \right\rangle \mid X_i^{i-1} \right\}
\]

+ \frac{1}{n} \sum_{i=1}^{n} \left( \ln \left\langle b^*(X_i^{i-1}), X_i \right\rangle - \mathbb{E}\left\{ \ln \left\langle b^*(X_i^{i-1}), X_i \right\rangle \mid X_i^{i-1} \right\} \right)
These limit relations give rise to the following definition:

**Definition**

A portfolio strategy $B$ is called **universally consistent with respect to a class $\mathcal{C}$ of stationary and ergodic processes $\{X_n\}_{-\infty}^{\infty}$**, if for each process in the class,

$$\lim_{n \to \infty} \frac{1}{n} \ln S_n(B) = W^*$$

almost surely.
Empirical portfolio selection

$$\mathbb{E}\{\ln \langle b^*(X_1^{n-1}), X_n \rangle | X_1^{n-1} \} = \max_{b(\cdot)} \mathbb{E}\{\ln \langle b(X_1^{n-1}), X_n \rangle | X_1^{n-1} \}$$
Empirical portfolio selection

\[ E\{\ln \langle b^*(X_1^{n-1}), X_n \rangle | X_1^{n-1} \} = \max_{b(\cdot)} E\{\ln \langle b(X_1^{n-1}), X_n \rangle | X_1^{n-1} \} \]

\[ b^*(x_1^{n-1}) = \arg\max_{b(\cdot)} E\{\ln \langle b(X_1^{n-1}), X_n \rangle | X_1^{n-1} = x_1^{n-1} \} \]
\[ \mathbb{E}\{\ln \langle b^{*}(X_{1}^{n-1}), X_{n} \rangle \mid X_{1}^{n-1} \} = \max_{b(\cdot)} \mathbb{E}\{\ln \langle b(X_{1}^{n-1}), X_{n} \rangle \mid X_{1}^{n-1} \} \]

\[ b^{*}(x_{1}^{n-1}) = \arg \max_{b(\cdot)} \mathbb{E}\{\ln \langle b(X_{1}^{n-1}), X_{n} \rangle \mid X_{1}^{n-1} = x_{1}^{n-1} \} \]

\[ = \arg \max_{b(\cdot)} \mathbb{E}\{\ln \langle b(x_{1}^{n-1}), X_{n} \rangle \mid X_{1}^{n-1} = x_{1}^{n-1} \} \]
\[
E\{\ln \langle b^*(X_1^{n-1}), X_n \rangle \mid X_1^{n-1}\} = \max_{b(\cdot)} E\{\ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1}\}
\]

\[
b^*(x_1^{n-1}) = \arg \max_{b(\cdot)} E\{\ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} = x_1^{n-1}\}
\]

\[
= \arg \max_{b(\cdot)} E\{\ln \langle b(x_1^{n-1}), X_n \rangle \mid X_1^{n-1} = x_1^{n-1}\}
\]

\[
= \arg \max_{b(\cdot)} E\{\ln \langle b, X_n \rangle \mid X_1^{n-1} = x_1^{n-1}\},
\]
Empirical portfolio selection

\[ E\{\ln \langle b^*(X_{1}^{n-1}), X_n \rangle \mid X_{1}^{n-1} \} = \max_{b(\cdot)} E\{\ln \langle b(X_{1}^{n-1}), X_n \rangle \mid X_{1}^{n-1} \} \]

\[ b^*(x_{1}^{n-1}) = \arg \max_{b(\cdot)} E\{\ln \langle b(X_{1}^{n-1}), X_n \rangle \mid X_{1}^{n-1} = x_{1}^{n-1} \} \]

\[ = \arg \max_{b(\cdot)} E\{\ln \langle b(x_{1}^{n-1}), X_n \rangle \mid X_{1}^{n-1} = x_{1}^{n-1} \} \]

\[ = \arg \max_{b} E\{\ln \langle b, X_n \rangle \mid X_{1}^{n-1} = x_{1}^{n-1} \}, \]

fixed integer \( k > 0 \)

\[ E\{\ln \langle b(X_{1}^{n-1}), X_n \rangle \mid X_{1}^{n-1} \} \approx E\{\ln \langle b(X_{n-k}^{n-1}), X_n \rangle \mid X_{n-k}^{n-1} \} \]
Empirical portfolio selection

\[ E\{\ln \langle b^*(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} \} = \max_{b(\cdot)} E\{\ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} \} \]

\[ b^*(x_1^{n-1}) = \arg \max_{b(\cdot)} E\{\ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} = x_1^{n-1} \} \]

\[ = \arg \max_{b(\cdot)} E\{\ln \langle b(x_1^{n-1}), X_n \rangle \mid X_1^{n-1} = x_1^{n-1} \} \]

\[ = \arg \max_{b(\cdot)} E\{\ln \langle b, X_n \rangle \mid X_1^{n-1} = x_1^{n-1} \}, \]

fixed integer \( k > 0 \)

\[ E\{\ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} \} \approx E\{\ln \langle b(X_{n-k}^{n-1}), X_n \rangle \mid X_{n-k}^{n-1} \} \]

and

\[ b^*(X_1^{n-1}) \approx b_k(X_{n-k}^{n-1}) = \arg \max_{b(\cdot)} E\{\ln \langle b(X_{n-k}^{n-1}), X_n \rangle \mid X_{n-k}^{n-1} \} \]
because of stationarity

\[
\mathbf{b}_k(x^k_1) = \arg \max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \langle \mathbf{b}(\mathbf{X}^{n-1}_{n-k}), \mathbf{X}_n \rangle \mid \mathbf{X}^{n-1}_{n-k} = x^k_1\}
\]
because of stationarity

\[
\begin{align*}
\mathbf{b}_k(x^k_1) &= \arg \max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \langle \mathbf{b}(X_{n-k}^{n-1}) , X_n \rangle \mid X_{n-k}^{n-1} = x^k_1\} \\
&= \arg \max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \langle \mathbf{b}(x^k_1) , X_n \rangle \mid X_{n-k}^{n-1} = x^k_1\}
\end{align*}
\]
because of stationarity

\[ b_k(x_1^k) = \arg \max_{b(\cdot)} \mathbb{E}\{ \ln \langle b(X_{n-k}^{n-1}), X_n \rangle \mid X_{n-k}^{n-1} = x_1^k \} \]

\[ = \arg \max_{b(\cdot)} \mathbb{E}\{ \ln \langle b(x_1^k), X_n \rangle \mid X_{n-k}^{n-1} = x_1^k \} \]

\[ = \arg \max_{b(\cdot)} \mathbb{E}\{ \ln \langle b(x_1^k), X_{k+1} \rangle \mid X_1^k = x_1^k \} \]
because of stationarity

\[
\begin{align*}
\mathbf{b}_k(x^k_1) & = \arg \max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \left\langle \mathbf{b}(X_{n-k}^{n-1}) , X_n \right\rangle | X_{n-k}^{n-1} = x^k_1\} \\
& = \arg \max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \left\langle \mathbf{b}(x^k_1) , X_n \right\rangle | X_{n-k}^{n-1} = x^k_1\} \\
& = \arg \max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \left\langle \mathbf{b}(x^k_1) , X_{k+1} \right\rangle | X^k_1 = x^k_1\} \\
& = \arg \max_{\mathbf{b}} \mathbb{E}\{\ln \left\langle \mathbf{b} , X_{k+1} \right\rangle | X^k_1 = x^k_1\},
\end{align*}
\]
because of stationarity

\[
\begin{align*}
    \mathbf{b}_k(x^k_1) &= \arg \max_{\mathbf{b}(\cdot)} \mathbb{E}\{ \ln \langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}), \mathbf{X}_n \rangle \mid \mathbf{X}_{n-k}^{n-1} = x^k_1 \} \\
    &= \arg \max_{\mathbf{b}(\cdot)} \mathbb{E}\{ \ln \langle \mathbf{b}(x^k_1), \mathbf{X}_n \rangle \mid \mathbf{X}_{n-k}^{n-1} = x^k_1 \} \\
    &= \arg \max_{\mathbf{b}(\cdot)} \mathbb{E}\{ \ln \langle \mathbf{b}(x^k_1), \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k = x^k_1 \} \\
    &= \arg \max_{\mathbf{b}} \mathbb{E}\{ \ln \langle \mathbf{b}, \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k = x^k_1 \},
\end{align*}
\]

which is the maximization of the regression function

\[
m_{\mathbf{b}}(x^k_1) = \mathbb{E}\{ \ln \langle \mathbf{b}, \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k = x^k_1 \}
\]
Regression function

$Y$ real valued
$X$ observation vector

A Distribution-Free Theory of Nonparametric Regression, Springer-Verlag, New York.
Regression function

\( Y \) real valued  
\( X \) observation vector  
Regression function

\[ m(x) = \mathbb{E}\{ Y \mid X = x \} \]

i.i.d. data: \( D_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\} \)
Regression function

\( Y \) real valued
\( X \) observation vector
Regression function

\[
m(x) = \mathbb{E}\{Y \mid X = x\}
\]

i.i.d. data: \( D_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\} \)
Regression function estimate

\[
m_n(x) = m_n(x, D_n)
\]

Regression function

$Y$ real valued
$X$ observation vector
Regression function

$$m(x) = \mathbf{E}\{ Y \mid X = x \}$$

i.i.d. data: $D_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$
Regression function estimate

$$m_n(x) = m_n(x, D_n)$$

Kernel regression estimate with window kernel

Regression function

$Y$ real valued
$X$ observation vector
Regression function

$$m(x) = \mathbb{E}\{ Y \mid X = x \}$$

i.i.d. data: $D_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$
Regression function estimate

$$m_n(x) = m_n(x, D_n)$$

Kernel regression estimate with window kernel
Bandwidth $r > 0$
Regression function

\( Y \) real valued
\( X \) observation vector

Regression function

\[
m(x) = \mathbb{E}\{Y \mid X = x\}
\]

i.i.d. data: \( D_n = \{(X_1, Y_1), \ldots , (X_n, Y_n)\} \)

Regression function estimate

\[
m_n(x) = m_n(x, D_n)
\]

Kernel regression estimate with window kernel

Bandwidth \( r > 0 \)

\[
m_n(x) = \frac{\sum_{i=1}^{n} Y_i I_{\{\|x-X_i\|\leq r\}}}{\sum_{i=1}^{n} I_{\{\|x-X_i\|\leq r\}}}
\]


A Distribution-Free Theory of Nonparametric Regression, Springer-Verlag, New York.

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Empirical portfolio selections
Regression function

$Y$ real valued
$X$ observation vector

Regression function

$$m(x) = \mathbb{E}\{Y \mid X = x\}$$

i.i.d. data: $D_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$

Regression function estimate

$$m_n(x) = m_n(x, D_n)$$

Kernel regression estimate with window kernel

Bandwidth $r > 0$

$$m_n(x) = \frac{\sum_{i=1}^{n} Y_i I_{\{\|x-X_i\| \leq r\}}}{\sum_{i=1}^{n} I_{\{\|x-X_i\| \leq r\}}}$$

\[ X \sim X_1^k \]
\[ X \sim X_1^k \]
\[ Y \sim \ln \langle b, X_{k+1} \rangle \]
\[ X \sim X_1^k \]
\[ Y \sim \ln \langle b, X_{k+1} \rangle \]
\[ m(x) = E\{ Y \mid X = x \} \sim m_b(x_1^k) = E\{ \ln \langle b, X_{k+1} \rangle \mid X_1^k = x_1^k \} \]
choose the radius $r_{k,\ell} > 0$ such that for any fixed $k$,

$$\lim_{\ell \to \infty} r_{k,\ell} = 0.$$
choose the radius $r_{k,\ell} > 0$ such that for any fixed $k$,

$$\lim_{\ell \to \infty} r_{k,\ell} = 0.$$  

for $n > k + 1$, define the expert $b^{(k,\ell)}$ by

$$b^{(k,\ell)}(x_{1}^{n-1}) = \arg \max_{b} \sum_{\left\{k < i < n: \|x_{i-k}^{i-1} - x_{n-k}^{n-1}\| \leq r_{k,\ell}\right\}} \ln \langle b, x_{i} \rangle,$$

if the sum is non-void, and $b_{0} = (1/d, \ldots, 1/d)$ otherwise.
for fixed $k, \ell = 1, 2, \ldots,$

$\mathbf{B}(k, \ell) = \{b(k, \ell)(\cdot)\}$, are called elementary portfolios.
Combining elementary portfolios

for fixed $k, \ell = 1, 2, \ldots,$

$B^{(k,\ell)} = \{ b^{(k,\ell)}(\cdot) \}$, are called elementary portfolios

How to choose $k, \ell$

- small $k$ or large $r_{k,\ell}$: large bias
- large $k$ and small $r_{k,\ell}$: few matching, large variance
Combining elementary portfolios

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How to choose \( k, \ell \)

- small \( k \) or large \( r_{k,\ell} \): large bias
- large \( k \) and small \( r_{k,\ell} \): few matching, large variance

Machine learning: combination of experts

Exponential weighing

combine the elementary portfolio strategies $B^{(k,\ell)} = \{b_n^{(k,\ell)}\}$
Exponential weighing

combine the elementary portfolio strategies \( \mathbf{B}^{(k,\ell)} = \{ b_n^{(k,\ell)} \} \)
let \( \{ q_{k,\ell} \} \) be a probability distribution on the set of all pairs \((k, \ell)\)
such that for all \( k, \ell \), \( q_{k,\ell} > 0 \).
Exponential weighing

combine the elementary portfolio strategies \( B^{(k,\ell)} = \{ b_n^{(k,\ell)} \} \)

let \( \{ q_{k,\ell} \} \) be a probability distribution on the set of all pairs \((k, \ell)\) such that for all \( k, \ell \), \( q_{k,\ell} > 0. \)

put

\[
w_{n,k,\ell} = q_{k,\ell} S_{n-1}(B^{(k,\ell)})
\]
Exponential weighing

combine the elementary portfolio strategies $B^{(k,\ell)} = \{b_n^{(k,\ell)}\}$

let $\{q_{k,\ell}\}$ be a probability distribution on the set of all pairs $(k, \ell)$ such that for all $k, \ell$, $q_{k,\ell} > 0$.

put

$$w_{n,k,\ell} = q_{k,\ell} S_{n-1}(B^{(k,\ell)})$$

and

$$v_{n,k,\ell} = \frac{w_{n,k,\ell}}{\sum_{i,j} w_{n,i,j}}.$$
Exponential weighing

combine the elementary portfolio strategies $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}_n^{(k,\ell)}\}$

let $\{q_{k,\ell}\}$ be a probability distribution on the set of all pairs $(k, \ell)$ such that for all $k, \ell$, $q_{k,\ell} > 0$.

put

$$w_{n,k,\ell} = q_{k,\ell} S_{n-1}(\mathbf{B}^{(k,\ell)})$$

and

$$v_{n,k,\ell} = \frac{w_{n,k,\ell}}{\sum_{i,j} w_{n,i,j}}.$$ 

the combined portfolio $\mathbf{b}$:

$$\mathbf{b}_n(x_1^{n-1}) = \sum_{k,\ell} v_{n,k,\ell} \mathbf{b}_n^{(k,\ell)}(x_1^{n-1}).$$
$$S_n(B) = \prod_{i=1}^{n} \left\langle b_i(x_1^{i-1}), x_i \right\rangle$$
\[ S_n(B) = \prod_{i=1}^{n} \left\langle b_i(x_i^{i-1}), x_i \right\rangle \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k,\ell} w_{i,k,\ell} \left\langle b_i^{(k,\ell)}(x_i^{i-1}), x_i \right\rangle}{\sum_{k,\ell} w_{i,k,\ell}} \]
\[ S_n(B) = \prod_{i=1}^{n} \left\langle b_i(x_1^{i-1}), x_i \right\rangle \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k,\ell} w_{i,k,\ell} \left\langle b_i^{(k,\ell)}(x_1^{i-1}), x_i \right\rangle}{\sum_{k,\ell} w_{i,k,\ell}} \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k,\ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)}) \left\langle b_i^{(k,\ell)}(x_1^{i-1}), x_i \right\rangle}{\sum_{k,\ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)})} \]
\[ S_n(B) = \prod_{i=1}^{n} \left< b_i(x_1^{i-1}) , x_i \right> \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k,\ell} w_{i,k,\ell} \left< b_i^{(k,\ell)}(x_1^{i-1}) , x_i \right>}{\sum_{k,\ell} w_{i,k,\ell}} \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k,\ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)}) \left< b_i^{(k,\ell)}(x_1^{i-1}) , x_i \right>}{\sum_{k,\ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)})} \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k,\ell} q_{k,\ell} S_i(B^{(k,\ell)})}{\sum_{k,\ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)})} \]
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\[ = \prod_{i=1}^{n} \frac{\sum_{k, \ell} w_{i,k,\ell} \left\langle b_i^{(k,\ell)}(x_1^{i-1}), x_i \right\rangle}{\sum_{k, \ell} w_{i,k,\ell}} \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k, \ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)}) \left\langle b_i^{(k,\ell)}(x_1^{i-1}), x_i \right\rangle}{\sum_{k, \ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)})} \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k, \ell} q_{k,\ell} S_i(B^{(k,\ell)})}{\sum_{k, \ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)})} \]

\[ = \sum_{k, \ell} q_{k,\ell} S_n(B^{(k,\ell)}), \]
The strategy $\mathbf{B}$ then arises from weighing the elementary portfolio strategies $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}_n^{(k,\ell)}\}$ such that the investor’s capital becomes

$$S_n(\mathbf{B}) = \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}).$$
The kernel-based portfolio scheme is universally consistent with respect to the class of all ergodic processes such that
\[ E\{|\ln X(j)|\} < \infty, \text{ for } j = 1, 2, \ldots, d. \]


www.szit.bme.hu/~gyorfi/kernel.pdf
Proof

We have to prove that

\[ \liminf_{n \to \infty} W_n(B) = \liminf_{n \to \infty} \frac{1}{n} \ln S_n(B) \geq W^* \quad \text{a.s.} \]

W.l.o.g. we may assume \( S_0 = 1 \), so that

\[ W_n(B) = \frac{1}{n} \ln S_n(B) \]
Proof

We have to prove that

$$\liminf_{n \to \infty} W_n(B) = \liminf_{n \to \infty} \frac{1}{n} \ln S_n(B) \geq W^* \quad \text{a.s.}$$

W.l.o.g. we may assume $S_0 = 1$, so that

$$W_n(B) = \frac{1}{n} \ln S_n(B)$$

$$= \frac{1}{n} \ln \left( \sum_{k,\ell} q_{k,\ell} S_n(B^{(k,\ell)}) \right)$$
Proof

We have to prove that

\[ \lim_{n \to \infty} \inf W_n(\mathcal{B}) = \lim_{n \to \infty} \inf \frac{1}{n} \ln S_n(\mathcal{B}) \geq W^* \quad \text{a.s.} \]

W.l.o.g. we may assume \( S_0 = 1 \), so that

\[
W_n(\mathcal{B}) = \frac{1}{n} \ln S_n(\mathcal{B})
\]

\[
= \frac{1}{n} \ln \left( \sum_{k,\ell} q_{k,\ell} S_n(\mathcal{B}^{(k,\ell)}) \right)
\]

\[
\geq \frac{1}{n} \ln \left( \sup_{k,\ell} q_{k,\ell} S_n(\mathcal{B}^{(k,\ell)}) \right)
\]
Proof

We have to prove that

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\[ \geq \frac{1}{n} \ln \left( \sup_{k, \ell} q_{k, \ell} S_n(B^{(k, \ell)}) \right) \]

\[ = \frac{1}{n} \sup_{k, \ell} \left( \ln q_{k, \ell} + \ln S_n(B^{(k, \ell)}) \right) \]

Györfi  
Empirical portfolio selections
We have to prove that

\[ \liminf_{n \to \infty} W_n(B) = \liminf_{n \to \infty} \frac{1}{n} \ln S_n(B) \geq W^* \quad \text{a.s.} \]

W.l.o.g. we may assume \( S_0 = 1 \), so that

\[ W_n(B) = \frac{1}{n} \ln S_n(B) \]

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\[ = \sup_{k, \ell} \left( W_n(B^{(k, \ell)}) + \frac{\ln q_{k, \ell}}{n} \right). \]
Thus

\[ \liminf_{n \to \infty} W_n(B) \geq \liminf_{n \to \infty} \sup_{k,\ell} \left( W_n(B^{(k,\ell)}) + \frac{\ln q_{k,\ell}}{n} \right) \]
Thus

\[\liminf_{n \to \infty} W_n(B) \geq \liminf_{n \to \infty} \sup_{k, \ell} \left( W_n(B^{(k, \ell)}) + \frac{\ln q_{k, \ell}}{n} \right)\]

\[\geq \sup_{k, \ell} \liminf_{n \to \infty} \left( W_n(B^{(k, \ell)}) + \frac{\ln q_{k, \ell}}{n} \right)\]

Because of \(\lim_{\ell \to \infty} r_{k, \ell} = 0\), we have that

\[\sup_{k, \ell} \liminf_{n \to \infty} W_n(B^{(k, \ell)}) = W^*.\]
Thus

\[
\liminf_{n \to \infty} W_n(B) \geq \liminf_{n \to \infty} \sup_{k, \ell} \left( W_n(B^{(k, \ell)}) + \frac{\ln q_{k, \ell}}{n} \right)
\]

\[
\geq \sup_{k, \ell} \liminf_{n \to \infty} \left( W_n(B^{(k, \ell)}) + \frac{\ln q_{k, \ell}}{n} \right)
\]

\[
= \sup_{k, \ell} \liminf_{n \to \infty} W_n(B^{(k, \ell)})
\]
Thus

\[
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\]

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\]

\[
= \sup_{k, \ell} \liminf_{n \to \infty} W_n(B^{(k,\ell)})
\]

\[
= \sup_{k, \ell} \epsilon_{k,\ell}
\]

Because of \( \lim_{\ell \to \infty} r_{k,\ell} = 0 \), we have that

\[
\sup_{k, \ell} \epsilon_{k,\ell} = \lim_{k \to \infty} \lim_{l \to \infty} \epsilon_{k,\ell} = W^*.
\]
Semi-log-optimal portfolio

empirical log-optimal:

\[ b^{(k,\ell)}(x_1^{n-1}) = \arg \max_b \sum_{i \in J_n} \ln \langle b, x_i \rangle \]
Semi-log-optimal portfolio

empirical log-optimal:

\[ \mathbf{b}^{(k,\ell)}(x_1^{n-1}) = \arg \max_{\mathbf{b}} \sum_{i \in J_n} \ln \langle \mathbf{b}, \mathbf{x}_i \rangle \]

Taylor expansion: \( \ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2 \)
empirical log-optimal:

$$b^{(k,\ell)}(x_1^{n-1}) = \arg \max_b \sum_{i \in J_n} \ln \langle b, x_i \rangle$$

Taylor expansion: $\ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2$ empirical semi-log-optimal:

$$\tilde{b}^{(k,\ell)}(x_1^{n-1}) = \arg \max_b \sum_{i \in J_n} h(\langle b, x_i \rangle) = \arg \max_b \{\langle b, m \rangle - \langle b, Cb \rangle\}$$
Semi-log-optimal portfolio

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Connection to the Markowitz theory.
Semi-log-optimal portfolio

empirical log-optimal:

\[ b^{(k,\ell)}(x_1^{n-1}) = \arg \max_b \sum_{i \in J_n} \ln \langle b, x_i \rangle \]

Taylor expansion: \( \ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2 \) empirical semi-log-optimal:

\[ \tilde{b}^{(k,\ell)}(x_1^{n-1}) = \arg \max_b \sum_{i \in J_n} h(\langle b, x_i \rangle) = \arg \max_b \{ \langle b, m \rangle - \langle b, Cb \rangle \} \]

Connection to the Markowitz theory.

Conditions of the model:

Assume that

- the assets are arbitrarily divisible,
- the assets are available in unbounded quantities at the current price at any given trading period,
- there are no transaction costs,
- the behavior of the market is not affected by the actions of the investor using the strategy under investigation.
At www.szit.bme.hu/~oti/portfolio there are two benchmark data set from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years.
- The second data set contains 23 stocks and has length 44 years.
At www.szit.bme.hu/~oti/portfolio there are two benchmark data set from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years.
- The second data set contains 23 stocks and has length 44 years.

Our experiment is on the second data set such that left four assets (SHERW, KODAK, COMME, KINAR) having small capitalization (less than $10^{10}$ dollars)
Kernel based semi-log-optimal portfolio selection with $k = 1, \ldots, 5$ and $l = 1, \ldots, 10$

$$r_{k,l}^2 = 0.0002 \cdot d \cdot k + 0.00002 \cdot d \cdot k \cdot \ell,$$
Experiments on average annual yields (AAY)

Kernel based semi-log-optimal portfolio selection with $k = 1, \ldots, 5$ and $l = 1, \ldots, 10$

$$r^2_{k,l} = 0.0002 \cdot d \cdot k + 0.00002 \cdot d \cdot k \cdot ℓ,$$

AAY of kernel based semi-log-optimal portfolio is 31%
Experiments on average annual yields (AAY)

Kernel based semi-log-optimal portfolio selection with $k = 1, \ldots, 5$ and $l = 1, \ldots, 10$

$$r_{k,l}^2 = 0.0002 \cdot d \cdot k + 0.00002 \cdot d \cdot k \cdot l,$$

AAY of kernel based semi-log-optimal portfolio is 31%.

MORRIS had the best AAY, 20%.
Kernel based semi-log-optimal portfolio selection with $k = 1, \ldots, 5$ and $l = 1, \ldots, 10$

$$r_{k,l}^2 = 0.0002 \cdot d \cdot k + 0.00002 \cdot d \cdot k \cdot l,$$

AY of kernel based semi-log-optimal portfolio is 31%
MORRIS had the best AAY, 20%
the BCRP had average AAY 21%
The average annual yields of the individual experts, for the 19 large assets.

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