Convergence of Natural Game Dynamics

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Outline

- Equilibria and Game Dynamics
- Convergence to Equilibria
  - Nash Dynamics
  - Regret-Minimization Dynamics
- Convergence to Nearly-Optimal Solutions
  - Nash Dynamics
  - Regret-Minimization Dynamics
- Conclusion
Games

Game:

- agents $\mathcal{N} = \{1, \ldots, n\}$
- $\forall i \in \mathcal{N}$: finite strategy space $\Sigma_i$
- $\forall i \in \mathcal{N}$: cost function $c_i: \Sigma_1 \times \cdots \times \Sigma_n \rightarrow \mathbb{R}$
  
  ($S \in \Sigma_1 \times \cdots \times \Sigma_n$ is called state.)
Games

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- agents $\mathcal{N} = \{1, \ldots, n\}$
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($S \in \Sigma_1 \times \cdots \times \Sigma_n$ is called state.)

Example: Network Congestion Games
Games

Game:
- agents $\mathcal{N} = \{1, \ldots, n\}$ drivers
- $\forall i \in \mathcal{N}$: finite strategy space $\Sigma_i$ possible paths from $s_i$ to $t_i$
- $\forall i \in \mathcal{N}$: cost function $c_i : \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{R}$ travel time
  ($S \in \Sigma_1 \times \cdots \times \Sigma_n$ is called state.)

Example: Network Congestion Games
latency function $\ell_e : \mathbb{N} \to \mathbb{R}$ for every edge $e$

$$\begin{align*}
  c_1(S) &= 8 \\
  c_2(S) &= 8 \\
  c_3(S) &= 2
\end{align*}$$
Games

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$$c_1(S) = 8 \quad c_2(S) = 8 \quad c_3(S) = 2$$

We consider only games with complete information.
Nash Equilibria

\[ c_1(S) = 4 \]
\[ c_2(S) = 1 \]
\[ c_3(S) = 5 \]

Definition

pure Nash Equilibrium \( S \in \Sigma_1 \times \cdots \times \Sigma_n \)
\[ \iff \] no player can unilaterally improve his payoff in \( S \)
Nash Equilibria

\[ c_1(S) = 3 \]
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Definition

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- Nash Equilibrium = stable
  (if players are uncoordinated, rational, selfish)
- We do not consider mixed Nash equilibria in this tutorial.
Properties of Equilibria

A lot of research on static properties of equilibria: How much does society suffer from selfish behavior?

- Let $\text{cost}$ be some measure for social cost, e.g.,
  
  $\text{cost}(S) = \sum_{i \in N} c_i(S)$ or $\text{cost}(S) = \max_{i \in N} c_i(S)$.

- price of anarchy $= \max_{S \in \text{NE}} \frac{\text{cost}(S)}{\text{cost}(\text{Opt})}$
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- \[
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Focus of this tutorial: **Questions about dynamics**

- Do uncoordinated agents reach an equilibrium?
- How long does it take?
- Do they quickly reach a state with small social cost?
Congestion Games

Congestion Game:

- set of players $\mathcal{N}$
- set of resources $\mathcal{R}$
  e.g., edges of a graph or set of servers

- set of strategies, $\forall i \in \mathcal{N} : \Sigma_i \subseteq 2^\mathcal{R}$
**Congestion Games**

**Congestion Game:**
- set of **players** $\mathcal{N}$
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- set of **strategies**,
  - $\forall i \in \mathcal{N} : \Sigma_i \subseteq 2^\mathcal{R}$
  - $\Sigma_i = \{ P \subseteq \mathcal{R} \mid P \text{ path } s_i \rightarrow t_i \}$ (network congestion game)
  - $\Sigma_i = \{ P \subseteq \mathcal{R} \mid P \text{ path } s \rightarrow t \}$ (symmetric congestion game)
  - $\Sigma_i = \{ \{ r \} \mid r \in \mathcal{R} \}$ (singleton congestion game)

- **latency functions**
  - $\forall r \in \mathcal{R} : \ell_r : \mathbb{N} \rightarrow \mathbb{N}$
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Nash Dynamics

- Nash Dynamics: Sequence of best responses of players.

$c_1(S) = 8$
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Nash Dynamics with Liveness Property: Each player gets a chance to play his/her best response after at most $t$ steps.

Random Nash Dynamics: Players are chosen uniformly at random.

$\epsilon$-Nash Dynamics: Players change their strategy only if they can improve their own cost by a factor of at least $1 + \epsilon$.

Other dynamics are discussed later.
Nash Dynamics

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The State Graph

**state graph** $\mathcal{G} = (V, E)$

$V = \text{states} \quad E = \text{better/best responses}$

There is an edge from state $S$ to $S'$ with label $i$ if player $i$ improves her cost from $S$ to $S'$. 
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Properties of dynamics can be phrased in terms of state graph:

- Pure Nash equilibrium = sink nodes of state graph
- Potential game = acyclic state graph $\Rightarrow$ players eventually reach equilibrium.

Example: Congestion Games
- Non-potential games = best responses may cycle.
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Equilibria and Game Dynamics

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    - Potential Games
    - NonPotential Games
  - Regret-Minimization Dynamics

Convergence to Nearly-Optimal Solutions
  - Nash Dynamics
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Conclusion
Rosenthal’s Potential Function for Congestion Games

Every congestion game admits an exact potential function.

\[ \Phi : \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{N} \text{ with } 0 \leq \Phi \leq n \cdot m \cdot d_{\text{max}} \]

- player decreases his delay by \( x \in \mathbb{N} \) \( \Rightarrow \) \( \Phi \) decreases by \( x \) as well

Rosenthal (Int. Journal of Game Theory 1973)
Rosenthal’s Potential Function for Congestion Games

\[ \phi(S) = 2 + 2 + (1 + 8) = 13 \]

\[ \phi(S') = 2 + (2 + 3) + 1 + 1 = 9 \]

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- player decreases his delay by \( x \in \mathbb{N} \) \( \Rightarrow \) \( \Phi \) decreases by \( x \) as well
- \( n_r = \) number of players \( i \) with \( r \in S_i \in \Sigma_i \)

\[ \phi(S) = \sum_{r \in \mathcal{R}} \sum_{i=1}^{n_r} d_r(i) \]
Rosenthal’s Potential Function for Congestion Games

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\phi(S) = 2 + 2 + (1 + 8) = 13
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\phi(S) = \sum_{r \in \mathcal{R}} \sum_{i=1}^{n_r} d_r(i)
\]

\( \Rightarrow \) The state graph is acyclic.
Known Results on Convergence Time

Fabrikant, Papadimitriou, Talwar (STOC 04)

There exist network congestion games with an initial state from which all better response sequences have exponential length.
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In singleton games all best response sequences have length at most $n^2 \cdot m$. 
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In singleton games all best response sequences have length at most $n^2 \cdot m$.

Ackermann, Röglin, Vöcking (FOCS 06)

- In spanning tree congestion games all best response sequences have length at most $n^2 \cdot m \cdot$ number of vertices.
- In matroid congestion games all best response sequences have length at most $n^2 \cdot m \cdot$ rank.
Singleton Games

- **Idea:** Reduce delays without affecting the game!

\[ d_r(n_r) > d_{r'}(n_{r'} + 1) \]
Singleton Games

- **Idea:** Reduce delays without affecting the game!
- **equivalent delays** $\overline{d}_r(x) \leq n \cdot m$

\[ \forall r, r' \in \mathcal{R}, n_r, n_{r'} : \]
\[ d_r(n_r) > d_{r'}(n_{r'} + 1) \]

\[ \iff \overline{d}_r(n_r) > \overline{d}_{r'}(n_{r'} + 1) \]

$\overline{d}_r(n_r) > \overline{d}_{r'}(n_{r'} + 1)$

However, delay reduction works also for matroid games.
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Network Congestion Games

$$d_{r_1}(n_{r_1}) + d_{r_2}(n_{r_2}) > d_{r'_1}(n_{r'_1} + 1) + d_{r'_2}(n_{r'_2} + 1)$$

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Singleton Games

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\forall r, r' \in \mathcal{R}, n_r, n_{r'} : \quad d_r(n_r) > d_{r'}(n_{r'} + 1) \\
\iff \overline{d}_r(n_r) > \overline{d}_{r'}(n_{r'} + 1)
\]

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d_{r_1}(n_{r_1}) + d_{r_2}(n_{r_2}) > d_{r_1'}(n_{r_1'} + 1) + d_{r_2'}(n_{r_2'} + 1)
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However, delay reduction works also for matroid games.
PLS: Polynomial Local Search Problems

Local Search Problem $\Pi$

- set of instances $\mathcal{I}_\Pi$
- for $I \in \mathcal{I}_\Pi$: set of feasible solutions $\mathcal{F}(I)$
- for $I \in \mathcal{I}_\Pi$: objective function $c: \mathcal{F}(I) \rightarrow \mathbb{Z}$
- for $I \in \mathcal{I}_\Pi$ and $S \in \mathcal{F}(I)$: neighborhood $\mathcal{N}(S, I) \subseteq \mathcal{F}(I)$
PLS: Polynomial Local Search Problems

Local Search Problem Π

- set of instances \( I_\Pi \)
- for \( I \in I_\Pi \): set of feasible solutions \( F(I) \)
- for \( I \in I_\Pi \): objective function \( c: F(I) \to \mathbb{Z} \)
- for \( I \in I_\Pi \) and \( S \in F(I) \): neighborhood \( N(S, I) \subseteq F(I) \)

Johnson, Papadimitriou, Yannakakis (FOCS 85)

Π is in **PLS** if polynomial time algorithms exist for

- finding **initial feasible solution** \( S \in F(I) \),
- computing the **objective value** \( c(S) \),
- finding a **better solution in the neighborhood** \( N(S, I) \) if \( S \) is not locally optimal.
PLS-reductions

**PLS-reduction**

- Polynomial-time computable function \( f : \mathcal{I}_{\Pi_1} \rightarrow \mathcal{I}_{\Pi_2} \).
- Polynomial-time computable function \((S_2 \in \mathcal{F}(f(I)))\) \( g : S_2 \mapsto S_1 \in \mathcal{F}(I) \)

\[ \begin{array}{c}
\mathcal{I}_{\Pi_1} \\
\mathcal{F}(I) \\
\end{array} \xrightarrow{f} \begin{array}{c}
\mathcal{I}_{\Pi_2} \\
\mathcal{F}(f(I)) \\
\end{array} \]

\[ \begin{array}{c}
\Pi_1 \\\n\Pi_2 \\
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  \[ g: S_2 \mapsto S_1 \in \mathcal{F}(I) \]
- $S_2$ locally optimal $\Rightarrow g(S_2)$ locally optimal.

- Tight reduction implies exponential running time.
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- local opt. of \( \Pi_2 \) easy to find \( \Rightarrow \) local opt. of \( \Pi_1 \) easy to find
- local opt. of \( \Pi_1 \) hard to find \( \Rightarrow \) local opt. of \( \Pi_2 \) hard to find

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Network Congestion Games and PLS

Fabrikant, Papadimitriou, Talwar (STOC 04), Ackermann, Röglin, Vöcking (FOCS 06)

Network congestion games are PLS-complete for (un)directed networks with linear delay functions.
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Network congestion games are PLS-complete for (un)directed networks with linear delay functions.

⇒ Computing a pure NE is hard.
Also, the PLS-reduction is tight.
⇒ There exist states exponentially far from all sinks in the state graph.
Approximate Equilibria

What happens if players are lazy?

A state $S = (S_1, \ldots, S_n)$ is called $(1 + \varepsilon)$-approximate equilibrium if $\forall i \in \mathcal{N}$: delay of player $i \leq (1 + \varepsilon) \cdot \min \text{achievable delay of player } i$
What happens if players are lazy?

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Positive Result:

Chien, Sinclair (SODA 07)

In any symmetric congestion game with $\alpha$-bounded jump condition, the $(1 + \varepsilon)$-Nash dynamics converges after at most $\text{poly}(n, \alpha, \varepsilon^{-1}, \log(d_{\text{max}}))$ steps, assuming liveness property.
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Idea: high-cost player moves $\Rightarrow$ significant potential drop

$S$ not $(1 + \varepsilon)$-equilibrium $\Rightarrow$ $\exists$ high-cost player that has an incentive to move. (due to $\alpha$-bounded jump condition and symmetry)
Approximate Equilibria

What happens if players are lazy?

Approximate Equilibria

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Negative Result:

Skopalik, Vöcking (STOC 2008)

It is PLS-hard to compute an \((1+\varepsilon)\)-approximate equilibrium for any polynomial-time computable \( \varepsilon \).

\( \Rightarrow \) Exponentially many steps until \((1+\varepsilon)\)-approx. eq. is reached. Very involved reduction from Circuit/Flip.
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Non-potential Games

**Sink equilibrium:** strongly connected comp. of state graph w/o outgoing edges [Goemans, M., Vetta]

⇒ random Nash dynamics eventually reaches sink equilibrium
Non-potential Games

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**Interesting class:** Games with only singleton sink equilibria

**Example:** player-specific singleton congestion games.

Milchtaich, Games and Economics Behaviour, 1996

In player-specific singleton congestion games the best-response dynamics can cycle. From every state there is a sequence of best-responses to a pure equilibrium.
How to find a stable marriage?

Let’s get to the really important problems...
The Stable Marriage Problem

Set of women $X$

Set of men $Y$
The Stable Marriage Problem

Set of women $\mathcal{X}$

( )

( )

( )

( )

Set of men $\mathcal{Y}$

( )

( )

( )

( )

Every person has a preference list.
The Stable Marriage Problem

Set of women \( \mathcal{X} \)

\[
(\text{green, blue, red, green})
\]

\[
(\text{green, blue, red, green})
\]

\[
(\text{green, blue, red, green})
\]

\[
(\text{green, blue, red, green})
\]

Set of men \( \mathcal{Y} \)

\[
(\text{red, green, blue, red})
\]

\[
(\text{red, green, blue, red})
\]

\[
(\text{red, green, blue, red})
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The Stable Marriage Problem

Set of women $X$

Set of men $Y$

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Formal Definition

Stable Matching

A matching is stable if there does not exist a blocking pair.

Theorem [Gale, Shapley 1962]
A stable matching can be computed efficiently.
**Stable Matching**

A matching is stable if there does not exist a blocking pair.

$(w, m')$ is blocking pair

1) $w$ prefers $m'$ to $m$
2) $m'$ prefers $w$ to $w'$

**Theorem [Gale, Shapley 1962]**

A stable matching can be computed efficiently.
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Stable Matching
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\[\iff\]
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Formal Definition

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\[(w, m')\text{ is blocking pair} \iff \]
\[1) \ w \text{ prefers } m' \text{ to } m \]
\[2) \ m' \text{ prefers } w \text{ to } w' \]

Theorem [Gale, Shapley 1962]

A stable matching can be computed efficiently.
Applications and Previous Work

- Many Applications: Interns/Hospitals, College Admission, Labor market.

  Many further results since the 60s: roommates, ties, incomplete preferences, many-to-many matchings, etc.

  Mechanism Design Questions: Can players benefit from lying? (Roth 1982)

Main Question

- What happens without central authority?
- Do players reach a stable matching?
- How long does it take?
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Best Response Dynamics

Matching not stable \(\Rightarrow\) Choose woman, let her play best response.
Best Response Dynamics

Matching **not stable** ⇒ Choose **woman**, let her play **best response**.
Best Response Dynamics

Matching not stable $\Rightarrow$ Choose woman, let her play best response.
Best Response Dynamics

Matching not stable ⇒ Choose woman, let her play best response.
Good news:

**Theorem**
From every matching there exists a sequence of $2n^2$ best responses to a stable matching.

⇒ Random best-response dynamics reaches a stable matching with probability 1.
Best Response Dynamics – Good News

Theorem
From every matching there exists a sequence of $2n^2$ best responses to a stable matching.

Claim 1
If only married women play best responses, after at most $n^2$ steps every married woman is happy.

Claim 2
If every married woman is happy, every sequence of best responses terminates after at most $n^2$ steps.
Claim 1
If only married women play best responses, after at most $n^2$ steps every married woman is happy.

Proof.
Use the following potential function:

$$\Phi = \sum_{\text{married woman } w} \text{rank of } w\text{'s current partner}$$

$0 \leq \Phi \leq n^2$ and $\Phi$ decreases with every best response.
Best Response Dynamics – Good News

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$$\Phi = 4$$
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$$

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1 \quad \circ\quad \circ \quad 1 \quad \circ\quad \circ \quad \Phi = 4

3 \quad \circ\quad \circ \quad \quad \rightarrow \quad 2 \quad \circ\quad \circ

\Phi = 3
Claim 1
If only married women play best responses, after at most $n^2$ steps every married woman is happy.

Proof.
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$$

$0 \leq \Phi \leq n^2$ and $\Phi$ decreases with every best response.

\[\begin{array}{c}
1 & \rightarrow & 1 \\
3 & \rightarrow & 2 \\
\Phi = 4 & \rightarrow & \Phi = 3
\end{array}\]
Claim 1
If only married women play best responses, after at most \( n^2 \) steps every married woman is happy.

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Use the following potential function:

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\]

\( 0 \leq \Phi \leq n^2 \) and \( \Phi \) decreases with every best response.
Claim 2
If every married woman is happy, every sequence of best responses terminates after at most \( n^2 \) steps.

Proof.
Invariant: No married woman can improve.
Claim 2
If every married woman is happy, every sequence of best responses terminates after at most $n^2$ steps.

Proof.
Invariant: No married woman can improve.
⇒ Men are never dumped.
Claim 2
If every married woman is happy, every sequence of best responses terminates after at most $n^2$ steps.

Proof.
Invariant: No married woman can improve. 
⇒ Men are never dumped.
Use the following potential function:

$$
\Psi = \sum_{\text{married man } m} (n + 1 - \text{rank of } m\text{'s current partner})
$$

$0 \leq \Psi \leq n^2$ and $\Psi$ increases with every best response.
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\[\Psi = 5\]
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Use the following potential function:

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\Psi = \sum_{\text{married man } m} n + 1 - \text{rank of } m\text{'s current partner}
$$

$0 \leq \Psi \leq n^2$ and $\Psi$ increases with every best response.

\[\begin{align*}
\Psi = 5 & \quad \rightarrow \quad \Psi = 6 \\
\Psi = 5 & \quad \rightarrow \quad \Psi = 5
\end{align*}\]
Claim 2
If every married woman is happy, every sequence of best responses terminates after at most \( n^2 \) steps.

Proof.
Invariant: No married woman can improve. 
\[ \Rightarrow \text{Men are never dumped.} \]
Use the following potential function:

\[ \Psi = \sum_{\text{married man } m} n + 1 - \text{rank of } m's \text{ current partner} \]

\(|0 \leq \Psi \leq n^2| \) and \( \Psi \) increases with every best response.
Lower Bound for Random Best Responses

Bad news:

**Theorem**
The best-response dynamics can cycle.
Bad news:

**Theorem**
The best-response dynamics can cycle.

**Theorem**
There exist instances such that the expected number of best responses is $\Omega(c^n)$ for some constant $c > 1$. 
Further Results – Correlated Instances

Good news: Correlation helps!

**Monotone Instances**
Input: complete, weighted bipartite graph $G = (V, E, w)$.
Every player tries to *maximize the weight* of her/his relationship.

```
3
2
2
1
```
Further Results – Correlated Instances

Good news: Correlation helps!

**Monotone Instances**
Input: complete, *weighted* bipartite graph \( G = (V, E, w) \).
Every player tries to maximize the weight of her/his relationship.

**Theorem**
Random best/better responses converge *in polynomial time* whp.
Outline

- Equilibria and Game Dynamics
- Convergence to Equilibria
  - Nash Dynamics
  - Regret-Minimization Dynamics
- Convergence to Nearly-Optimal Solutions
  - Nash Dynamics
  - Regret-Minimization Dynamics
- Conclusion
Natural Distributed/Synchronous Dynamics

- Fictitious Play
- Replicator dynamics
- No regret
Natural Distributed/Synchronous Dynamics

- **Fictitious Play**
  - play a best response to the empirical distribution of the opponents.
  - Nash equilibrium is an “absorbing state”

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  - Most reasonable variants converge in potential games [Sandholm JET 2001]
  - Convergence rate [Racke et al. STOC06]

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- **No regret**
  - Today
Natural Distributed/Synchronous Dynamics

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    [Sandholm JET 01]
  - Convergence rate [Racke et al. STOC 06]

- No regret

- Known to converge in specific games to Nash equilibrium

- There exist games on which uncoupled dynamics do not converge [Hart and Mas-Colell] a simple example for no regret [Zinkevich 03]
No regret in Congestion Games

- Is there a strategy that guarantees that the total routing time will take almost as time as the best fixed path in hindsight?
No regret in Congestion Games

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**No External Regret**
No regret in Congestion Games

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**No External Regret**

- Is there a strategy that guarantees that the total routing time when it took path P will take almost as time as the best fixed path in hindsight for that time steps?
No regret in Congestion Games

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No regret in Congestion Games

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**No External Regret**

- Is there a strategy that guarantees that the total routing time when it took path P will take almost as time as the best fixed path in hindsight for that time steps?

**No Internal Regret**

We say that algorithm is No X-Regret if its regret to best static decision, $R(T)$ is sublinear.
No Regret - Motivation

These properties can influence a rational user to adapt these algorithms (note that in stochastic setting these algorithms will converge to the optimal strategy)
No Regret convergence outline

- No internal regret convergence to Correlated equilibrium
No Regret convergence outline

- No internal regret convergence to Correlated equilibrium
- No external regret and zero sum games
No Regret convergence outline

- No internal regret convergence to Correlated equilibrium
- No external regret and zero sum games
- No external regret - non convergence examples
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- No external regret - non convergence examples
- No external regret and Routing games
- No external regret and socially concave games
Equilibria Types

- No Regret
- Correlated Equilibrium
- Mixed Nash Equilibrium
- Pure Nash Equilibrium
Correlated Equilibria [Aumann 1974]

- Distribution over $N$-tuples.
Correlated Equilibria [Aumann 1974]

- Distribution over $N$-tuples.
- Nash Equilibrium with a shared signal
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- **Distribution over** $N$-tuples.
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  - Private signal - not necessarily convex hull of Nash equilibrium (e.g. chicken game)
Correlated Equilibria [Aumann 1974]

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- Nash Equilibrium with a shared signal
  - Independent signal - Nash equilibrium
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  - Private signal - not necessarily convex hull of Nash equilibrium (e.g. chicken game)

Properties:

- Contains the convex hull of Nash equilibrium.
- Can be computed efficiently
[Hart and Mas-Colell, Foster and Vohra] If every player plays a no internal regret algorithm, then the empirical distributions of play converge almost surely as $t \to \infty$ to the set of correlated equilibrium distributions of the game.

The convergence is of the empirical distributions and not at a specific time.
No internal regret simple algorithm

Regret Matching [Hart and Mas-Collel]
No internal regret simple algorithm

Regret Matching [Hart and Mas-Collel]

- Inertia
No internal regret simple algorithm

Regret Matching [Hart and Mas-Collel]

- Inertia
- Switching probability

Computational side:
- All implementation requires space which is number of actions $2^n$
- No efficient implementation for continuous case
- Influence convergence rate as well.
No internal regret simple algorithm

Regret Matching [Hart and Mas-Collel]

- Inertia

- Switching probability
  - $R(i, k)$ - regret of not playing $k$ instead of $i$
  - Switching to action $j$ from action $i$ is proportional to $R(i, j)$
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There exists many others algorithms, see Foster and Vohra, Blum and Mansour, Lugosi and Stoltz ,...
No internal regret simple algorithm

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No external regret and online learning

- Number of mistakes
- Regret to a best hypothesis in a class
No external regret and online learning

How to measure online algorithms?
No external regret and online learning

How to measure online algorithms?

- Number of mistakes
- Regret to a best hypothesis in a class
Follow the Regularized Leader

Let $\ell_\tau$ be the loss function at time $\tau$

$$w_{t+1} = \arg\min_w \left[ \sum_{\tau=1}^{T} \eta \ell_\tau(w) + \text{Regulizer}(w) \right]$$
No external Regret - generic algorithm

Follow the Regularized Leader

Let $\ell_\tau$ be the loss function at time $\tau$

$$w_{t+1} = \arg\min_w \left[ \sum_{\tau=1}^{T} \eta \ell_\tau(w) + \text{Regulizer}(w) \right]$$

Includes, gradient descent, weight majority and more.
## No External Regret - History

### Evolution of Bounds

<table>
<thead>
<tr>
<th>Author and Year</th>
<th>Rate</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hannan 56</td>
<td>$O(\sqrt{NT})$</td>
<td>Adapted by KV</td>
</tr>
<tr>
<td>Blackwell 57</td>
<td>$O(\sqrt{NT})$</td>
<td>Sufficient conditions</td>
</tr>
<tr>
<td>Littlestone and Warmuth 89</td>
<td>$O(\sqrt{\log(N)T})$</td>
<td>weighted majority</td>
</tr>
<tr>
<td>Cesa Bianchi et al. 93</td>
<td>$O(\sqrt{\log(N)T})$</td>
<td>Optimal</td>
</tr>
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</table>
No External Regret - History

For the bandit setting

<table>
<thead>
<tr>
<th>Author</th>
<th>Setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lai and Robinns 85</td>
<td>Normal dist.</td>
</tr>
<tr>
<td>Auer et al. 95</td>
<td>$O(\log T)$</td>
</tr>
<tr>
<td>Bartlett et al. 08</td>
<td>$O(\sqrt{T})$</td>
</tr>
<tr>
<td>Aberenthy et al. 09</td>
<td>$O(\sqrt{T})$</td>
</tr>
</tbody>
</table>

More sets less efficient
Convex sets efficient

Applications to special cases

<table>
<thead>
<tr>
<th>Author</th>
<th>Settings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Helmbbold and Schapire</td>
<td>Prunning Decision trees</td>
</tr>
<tr>
<td>Takimoto and Warmuth</td>
<td>shortest path</td>
</tr>
<tr>
<td>Kalai and Vempala</td>
<td>Hannan’s algorithm for many settings</td>
</tr>
<tr>
<td>E. et al.</td>
<td>MDPs</td>
</tr>
<tr>
<td>Zinkevich</td>
<td>Convex functions</td>
</tr>
<tr>
<td>Aggarwarl at al</td>
<td>strongly convex function</td>
</tr>
<tr>
<td>Lugosi et al.</td>
<td>Bin Packing</td>
</tr>
<tr>
<td>E. et al</td>
<td>Load balancing</td>
</tr>
</tbody>
</table>
convergence in two players zero sum games

[Freund and Schapire Game and Economic Behavior 98]

- $M$ - the first player loss matrix.
- Minmax/Maxmin strategies: $p^*$, $q^*$
- $v$ value of the game, $M(p^*, q^*) = v$. 
convergence in two players zero sum games

[Freund and Schapire Game and Economic Behavior 98]

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Let $p_t, q_t$ be the strategies taken at time $t$:

$$\sum_{t=1}^{T} M(p_t, q_t) \leq$$
convergence in two players zero sum games

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\]
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\leq \sum_{t=1}^{T} M(p^*, q_t) + R(T) \leq T \cdot v + R(T)
$$
No External Regret and Routing Games

- Atomic games specific update rule [Kleinberg, Piliouras and Tardos STOC 09], Parallel links [Blum, E. and Ligett PODC 06]
- Splittable traffic [E., Mansour and Nadav STOC 09]
- Infinitesimal users (Wardrop model) [Blum, E. and Ligett PODC 06]
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A change in the model

- Infinitesimal users, assume over all traffic is 1
- All latency function $d_e(x)$ are non decreasing
- Multi commodity flow with $K$ types
Wardrop Model

\[ S_1 \]
\[ \ell(x) = 0 \]
\[ S_2 \]
\[ \ell(x) = 0 \]
\[ T \]
\[ \ell(x) = 0 \]
\[ \ell(x) = x \]
\[ \ell(x) = 2x \]
\[ \ell(x) = 0 \]
\[ \ell(x) = x \]
\[ \ell(x) = 2x \]
Convergence type

- $L_1$ convergence
- All users converge to a pure Nash equilibrium
Convergence type

- $L_1$ convergence
- All users converge to a pure Nash equilibrium

A flow $f$ is at equilibrium if and only if for every player type $i$, and paths $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $\ell_{P_1}(f) \leq \ell_{P_2}(f)$.

- $\mathcal{P}_i$ - possible paths for type $i$
- $f_{P_j}$ - flow on $P_j$

We would like both the average flow and the average cost to converges to Nash equilibrium
Convergence Theorems[Blum, E, Liggett]

Theorem

Let \( \epsilon' = \epsilon + 2\sqrt{s\epsilon n} \). Then for general functions with maximum slope \( s \), for \( T \geq T_\epsilon \), the time-average flow is \( \epsilon' \)-Nash, that is,

\[
\sum_{e \in E} l_e(\hat{f}_e)\hat{f}_e \leq \epsilon + 2\sqrt{s\epsilon n} + \sum_i a_i \min_{P \in \mathcal{P}_i} \sum_{e \in P} l_e(\hat{f}_e).
\]

Theorem

In general routing games with general delay functions with maximum slope \( s \), for all but a \( (ms^{1/4}\epsilon^{1/4}) \) fraction of time steps up to time \( T_\epsilon \), \( f^t \) is a \( (\epsilon + 2\sqrt{s\epsilon n} + 2m^{3/4}s^{1/4}\epsilon^{1/4}) \)-Nash flow.
Simple theorem and proof for linear latency functions

Theorem
Suppose the delay functions are linear. Then for $T \geq T_\epsilon$, the average flow $\hat{f}$ is $\epsilon$-Nash, i.e.

$$C(\hat{f}) \leq \epsilon + \sum_i a_i \min_{P \in \mathcal{P}_i} \sum_{e \in P} \ell_e(\hat{f}_e).$$
Simple proof for linear delay functions

Linearity:

- $\ell_e(\hat{f}_e) = \frac{1}{T} \sum_{t=1}^{T} \ell_e(f^t_e)$
- $\ell_e(f^t_e) f^t_e$ is convex
Simple proof for linear delay functions

**Linearity:**

- $\ell_e(\hat{f}_e) = \frac{1}{T} \sum_{t=1}^{T} \ell_e(f_t^e)$
- $\ell_e(f_t^e)f_t^e$ is convex

**Convexity:**

$$\ell_e(\hat{f}_e)\hat{f}_e \leq \frac{1}{T} \sum_{t=1}^{T} \ell_e(f_t^e) f_t^e.$$
Simple proof for linear delay functions

**Linearity:**

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**Convexity:**

$$\ell_e(\hat{f}_e)\hat{f}_e \leq \frac{1}{T} \sum_{t=1}^{T} \ell_e(f_t^t)f_t^t.$$ 

**Combining all:**

$$C(\hat{f}) \leq \frac{1}{T} \sum_{t=1}^{T} C(f_t^t) \leq \epsilon + \sum_i a_i \min_{P \in \mathcal{P}_i} \frac{1}{T} \sum_{t=1}^{T} \sum_{e \in P} \ell_e(f_t^t) \leq \epsilon + \sum_i a_i \min_{P \in \mathcal{P}_i} \sum_{e \in P} \ell_e(\hat{f}_e).$$
Socially concave games

A subclass of concave games [Rosen]

- There exists a combination \( \lambda_1, \ldots, \lambda_n \) such that
  \[
  \sum_{i=1}^{N} \lambda_i U_i(x) \text{ is concave}
  \]

- \( u_i(x_i, x_{-i}) \) is convex in \( x_{-i} \)

Theorem (E., Mansour and Nadav)

If every player in a socially concave games follows a no regret policy then:

- The average strategy vector converges to \( \epsilon \)-Nash equilibrium

- The average utility converges to the payoff at \( \epsilon \)-Nash equilibrium
Socially concave games

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- The average utility converges to the payoff at $\epsilon$-Nash equilibrium
Socially concave games

- Cournot competition (Best response does not converge)
Socially concave games

- Cournot competition (Best response does not converge)
- Resource allocation. (Best response does not converge)
Socially concave games

- Cournot competition \((\text{Best response does not converge})\)
- Resource allocation. \((\text{Best response does not converge})\)
- Atomic splittable routing
Socially concave games

- Cournot competition *(Best response does not converge)*
- Resource allocation. *(Best response does not converge)*
- Atomic splittable routing
- Congestion control protocols.
Outline

- Equilibria and Game Dynamics
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- Convergence to Nearly-Optimal Solutions
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Price of Anarchy and Convergence

- Price of anarchy = \( \frac{\text{Social Value of the worst equilibrium}}{\text{Optimal Social Value}} \).
Price of Anarchy and Convergence

- **Price of anarchy** = \( \frac{\text{Social Value of the worst equilibrium}}{\text{Optimal Social Value}} \).

- **Large** Price of Anarchy: Need for Central Regulation.
- **Small** Price of Anarchy: Does not indicate good performance.
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**Question 1:** Potential Games: How fast do players converge to approximate solutions? (and not to equilibria).

**Question 2:** Non-Potential Games: What is the quality of solutions that players converge to?
Congestion Games: Convergence to Nearly-Optimal Solutions

- **Question 1 (Potential Games):** How fast do players converge to approximate solutions? *(and not to equilibria).*
- **Price of anarchy:** 2.5 *(Koutsoupias, Christoudolou, 05 and Awerbuch, Azar, Epstein, 05).*
Congestion Games: Convergence to Nearly-Optimal Solutions

- **Question 1 (Potential Games):** How fast do players converge to approximate solutions? *(and not to equilibria).*
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How about convergence time to constant-factor approximate solutions?
Convergence to Nearly-optimal Solutions

- Theorem (Awerbuch, Azar, Epstein, M., Skopalik, EC 2008)
  Convergence time of Nash dynamics with liveness property to constant-factor optimal solutions in linear congestion games might be exponential.

- Proof Idea: Three lemmas:
  1. In any bad state, there exists a player who improves the average by a large margin, thus there is a state.
  2. In any bad state, the expected value of the change incurred by players is not too bad.
  3. Use induction on the above lemmas.

⇒ The price of anarchy for sink equilibrium is a constant.
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  For a large class of potential games that are $\beta$-nice, and satisfy bounded-jump condition, after polynomial steps of $\epsilon$-Nash dynamics with a liveness property, players converge to a solution with approximation factor of price of anarchy.

- Bounded-jump condition (informal): After a player $i$ plays a best response, the change in the payoff (cost) of other players is bounded by the new payoff (cost) of player $i$.

- For example:
  - Congestion games with constant-degree polynomial delay functions,
  - Weighted congestion games with linear delay functions,
  - Party affiliation games,
  - Market sharing games.
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## Summary of Convergence to Nearly-Optimal Solutions

Convergence to Nash equilibria: exponential

Convergence to nearly-optimal solutions:

<table>
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<tr>
<th>Game</th>
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For other games, check the $\beta$-nice and bounded jump condition.
Sink Equilibria and Convergence

- Question 2 (Non-Potential Games): What is the quality of solutions that players converge to?

- Price of anarchy for mixed NE might be good, but how about convergence to good-quality solutions in non-potential games?

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⇒ Players may converge to a bad-quality solution and they may get stuck there.

- What if players follow other dynamics?
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Quality of playing no regret

- In congestion games same bounds hold through similar arguments [Roughgarden STOC 09]
- Valid utility games and Hotelling games [Blum et al. STOC 08]
Quality of playing no regret

Recall

No Regret

Correlated Equilibrium

Mixed Nash Equilibrium

Pure Nash Equilibrium

price of No regret $\geq$ price of Correlated $\geq$ price of Mixed N.E $\geq$ price of Pure N.E
Quality of playing no regret

Recall

price of **No regret** \(\geq\) price of **Correlated** \(\geq\) price of **Mixed N.E** \(\geq\) price of **Pure N.E**
Load balancing example

Consider \( n \) parallel links and \( n \) identical users and Makespan metric then:

Pure N.E and sink : \( P_{ofA} = 1 \)
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Valid-Utility Games

Consider valid-utility games then:
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Pure N.E to No Regret: $\text{PofA} = 2$
Outline

- Equilibria and Game Dynamics
- Congestion Games
- Convergence to Equilibria
  - Nash Dynamics
  - Regret-Minimization Dynamics
- Convergence to Nearly-Optimal Solutions
  - Nash Dynamics
  - Regret-Minimization Dynamics
- Conclusion
In many realistic games learning algorithms can lead to Nash equilibrium or high quality state (later)
  ▶ Can be used to explain N.E
  ▶ Can be used for computing N.E

What can we say about games where nice behavior is not guaranteed?

Different types of regret for computing N.E in large games [Counterfactual, Zinkevich 07]

Effect of using machine learning algorithms and game dynamics in (ad) Auctions (or everywhere...)
Thank You

Special thanks to Heiko Roeglin for sharing his slides with us from another joint tutorial.