18.02 Multivariable Calculus
Fall 2007

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Spherical coordinates \((\rho, \phi, \theta)\).

\(\rho = \text{rho} = \text{distance to origin. } \phi = \text{phi} = \text{angle down from z-axis. } \theta = \text{same as in cylindrical coordinates. }\)

Diagrams drawn in space, and picture of 2D slice by vertical plane with \(z, r\) coordinates.

Formulas to remember: \(z = \rho \cos \phi, r = \rho \sin \phi\) (so \(x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta\)).

\(\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.\) The equation \(\rho = a\) defines the sphere of radius \(a\) centered at 0.

On the surface of the sphere, \(\phi\) is similar to latitude, except it’s 0 at the north pole, \(\pi/2\) on the equator, \(\pi\) at the south pole. \(\theta\) is similar to longitude.

\(\phi = \pi/4\) is a cone (asked using flash cards) \((z = r = \sqrt{x^2 + y^2}).\) \(\phi = \pi/2\) is the xy-plane.

**Volume element:** \(dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.\)

To understand this formula, first study surface area on sphere of radius \(a\): picture shown of a “rectangle” corresponding to \(\Delta \phi, \Delta \theta\), with sides = portion of circle of radius \(a\), of length \(a \Delta \phi\), and portion of circle of radius \(r = a \sin \phi\), of length \(r \Delta \theta = a \sin \phi \Delta \theta\). So \(\Delta S \approx a^2 \sin \phi \, \Delta \phi \Delta \theta\), which gives the surface element \(dS = a^2 \sin \phi \, d\phi d\theta\).

The volume element follows: for a small “box”, \(\Delta V = \Delta S \Delta \rho\), so \(dV = d\rho dS = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.\)

Example: recall the complicated example at end of Friday’s lecture (region sliced by a plane inside unit sphere). After rotating coordinate system, the question becomes: volume of the portion of unit sphere above the plane \(z = 1/\sqrt{2}\)? (picture drawn). This can be set up in cylindrical (left as exercise) or spherical coordinates.

For fixed \(\phi, \theta\) we are slicing our region by rays straight out of the origin; \(\rho\) ranges from its value on the plane \(z = 1/\sqrt{2}\) to its value on the sphere \(\rho = 1\). Spherical coordinate equation of the plane: \(z = \rho \cos \phi = 1/\sqrt{2}\), so \(\rho = \sec \phi / \sqrt{2}\). The volume is:

\[
\int_0^{2\pi} \int_0^{\pi/4} \int_0^{1/\sqrt{2} \sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta. 
\]

(Bound for \(\phi\) explained by looking at a slice by vertical plane \(\theta = \text{constant}\): the edge of the region is at \(z = r = 1/\sqrt{2}\)).

Evaluation: not done. Final answer: \(2\pi / 3 - 5\pi / 6\sqrt{2}\).

**Application to gravitation.**

Gravitational force exerted on mass \(m\) at origin by a mass \(\Delta M\) at \((x, y, z)\) (picture shown) is given by \(|\vec{F}| = G \Delta M m / \rho^2\), \(\operatorname{dir}(\vec{F}) = (x, y, z) / \rho\), i.e. \(\vec{F} = G \Delta M m / \rho^3 (x, y, z).\) (\(G\) is gravitational constant).

If instead of a point mass we have a solid with density \(\delta\), then we must integrate contributions to gravitational attraction from small pieces \(\Delta M = \delta \Delta V\). So

\[
\vec{F} = \iint_{R} \frac{Gm (x, y, z)}{\rho^3} \delta \, dV, \quad \text{i.e. } z\text{-component is } F_z = Gm \iiint_{R} \frac{z}{\rho^3} \delta \, dV, \ldots 
\]

If we can set up to use symmetry, then \(F_z\) can be computed nicely using spherical coordinates.

**General setup:** place the mass \(m\) at the origin (so integrand is as above), and place the solid so that the \(z\)-axis is an axis of symmetry. Then \(\vec{F} = (0, 0, F_z)\) by symmetry, and we have only one.
component to compute. Then
\[
F_z = Gm \iiint R \rho^2 \delta dV = Gm \iiint R \rho^2 \delta \rho^2 \sin \phi d\rho d\phi d\theta = Gm \iiint R \delta \cos \phi \sin \phi d\rho d\phi d\theta.
\]

Example: Newton’s theorem: the gravitational attraction of a spherical planet with uniform density \( \delta \) is the same as that of the equivalent point mass at its center.

\[
\text{Setup: the sphere has radius } a \text{ and is centered on the positive } z\text{-axis, tangent to } xy\text{-plane at the origin; the test mass is } m \text{ at the origin. Then}
\]
\[
F_z = Gm \iiint R \rho^2 \delta dV = Gm \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2a \cos \phi} \delta \cos \phi \sin \phi d\rho d\phi d\theta = \cdots = \frac{4}{3} Gm \delta \pi a = \frac{GMm}{a^2},
\]

where \( M = \text{mass of the planet} = \frac{4}{3} \pi a^3 \delta \). (The bounds for \( \rho \) and \( \phi \) need to be explained carefully, by drawing a diagram of a vertical slice with \( z \) and \( r \) coordinate axes, and the inscribed right triangle with vertices the two poles of the sphere + a point on its surface, the hypothenuse is the diameter \( 2a \) and we get \( \rho = 2a \cos \phi \) for the spherical coordinate equation of the sphere).]

18.02 Lecture 27. – Thu, Nov 15, 2007

Handouts: PS10 solutions, PS11

**Vector fields in space.**

At every point in space, \( \vec{F} = P \hat{i} + Q \hat{j} + R \hat{k} \), where \( P, Q, R \) are functions of \( x, y, z \).

Examples: force fields (gravitational force \( \vec{F} = -c(x, y, z)/\rho^2 \); electric field \( \vec{E} \), magnetic field \( \vec{B} \)); velocity fields (fluid flow, \( \vec{v} = \vec{v}(x, y, z) \)); gradient fields (e.g. temperature and pressure gradients).

**Flux.**

Recall: in 2D, flux of a vector field \( \vec{F} \) across a curve \( C = \int_C \vec{F} \cdot \hat{n} \, ds \).

In 3D, flux of a vector field is a double integral: flux through a surface, not a curve!

\( \vec{F} \) vector field, \( S \) surface, \( \hat{n} \) unit normal vector: \quad Flux = \oiint \vec{F} \cdot \hat{n} \, dS.

Notation: \( dS^\parallel = \hat{n} \, dS \). (We’ll see that \( dS^\parallel \) is often easier to compute than \( \hat{n} \) and \( dS \)).

Remark: there are 2 choices for \( \hat{n} \) (choose which way is counted positively!)

**Geometric interpretation of flux:**

As in 2D, if \( \vec{F} = \text{velocity of a fluid flow} \), then flux = flow per unit time across \( S \).

Cut \( S \) into small pieces, then over each small piece: what passes through \( \Delta S \) in unit time is the contents of a parallelepiped with base \( \Delta S \) and third side given by \( \vec{F} \).

Volume of box = base \( \times \) height = \( (\vec{F} \cdot \hat{n}) \Delta S \).

- **Examples:**
  1) \( \vec{F} = x \hat{i} + y \hat{j} + z \hat{k} \) through sphere of radius \( a \) centered at 0.
  \( \hat{n} = \frac{1}{a}(x, y, z) \) (other choice: \( -\frac{1}{a}(x, y, z) \); traditionally choose \( \hat{n} \) pointing out).
  \( \vec{F} \cdot \hat{n} = (x, y, z) \cdot \frac{1}{a}(x^2 + y^2 + z^2) = a \), so \( \oiint_S \vec{F} \cdot \hat{n} \, dS = \oiint_S a \, dS = a \, (4\pi a^2) \).
2) Same sphere, \( \vec{H} = z \hat{k} \): \( \vec{H} \cdot \hat{n} = \frac{z^2}{a} \).

\[
\iint_S \vec{H} \cdot d\vec{S} = \iint_S \frac{z^2}{a} \ dS = \int_0^{2\pi} \int_0^a a^2 \cos^2 \phi \ a^2 \sin \phi \ d\phi \ d\theta = 2\pi a^3 \int_0^\pi \cos^2 \phi \sin \phi \ d\phi = \frac{4}{3} \pi a^3.
\]

**Setup.** Sometimes we have an easy geometric argument, but in general we must compute the surface integral. The setup requires the use of two parameters to describe the surface, and \( \vec{F} \cdot \hat{n} \ dS \) must be expressed in terms of them. How to do this depends on the type of surface. For now, formulas to remember:

0) plane \( z = a \) parallel to \( xy \)-plane: \( \hat{n} = \pm \hat{k}, \ dS = dx \ dy. \) (similarly for planes // \( xz \) or \( yz \)-plane).

1) sphere of radius \( a \) centered at origin: use \( \phi, \theta \) (substitute \( \rho = a \) for evaluation); \( \hat{n} = \frac{1}{a} (x, y, z), \ dS = a^2 \sin \phi \ d\phi \ d\theta. \)

2) cylinder of radius \( a \) centered on \( z \)-axis: use \( z, \theta \) (substitute \( r = a \) for evaluation): \( \hat{n} \) is radially out in horizontal directions away from \( z \)-axis, i.e. \( \hat{n} = \frac{1}{a} (x, y, 0) \); and \( dS = a \ dz \ d\theta \) (explained by drawing a picture of a “rectangular” piece of cylinder, \( \Delta S = (\Delta z) (a \Delta \theta) \)).

3) graph \( z = f(x, y) \): use \( x, y \) (substitute \( z = f(x, y) \)). We’ll see on Friday that \( \hat{n} \) and \( dS \) separately are complicated, but \( \hat{n} \ dS = (-f_x, -f_y, 1) \ dx \ dy. \)

**18.02 Lecture 28. – Fri, Nov 16, 2007**

Last time, we defined the flux of \( \vec{F} \) through surface \( S \) as \( \iint \vec{F} \cdot \hat{n} \ dS \), and saw how to set up in various cases. Continue with more:

**Flux through a graph.** If \( S \) is the graph of some function \( z = f(x, y) \) over a region \( R \) of \( xy \)-plane: use \( x \) and \( y \) as variables. Contribution of a small piece of \( S \) to flux integral?

Consider portion of \( S \) lying above a small rectangle \( \Delta x \Delta y \) in \( xy \)-plane. In linear approximation it is a parallelogram. (picture shown)

The vertices are \((x, y, f(x, y))\); \((x + \Delta x, y, f(x + \Delta x, y))\); \((x, y + \Delta y, f(x, y + \Delta y))\); etc. Linear approximation: \( f(x + \Delta x, y) \approx f(x, y) + \Delta x f_x(x, y) \), and \( f(x, y + \Delta y) \approx f(x, y) + \Delta y f_y(x, y) \).

So the sides of the parallelogram are \((\Delta x, 0, \Delta x f_x)\) and \((0, \Delta y, \Delta y f_y)\), and

\[
\Delta \vec{S} = (\Delta x \langle 1, 0, f_x \rangle) \times (\Delta y \langle 0, 1, f_y \rangle) = \Delta x \Delta y \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1) \Delta x \Delta y.
\]

So \( d\vec{S} = \pm (-f_x, -f_y, 1) \ dx \ dy. \)

(From this we can get \( \hat{n} = \text{dir}(d\vec{S}) = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}} \) and \( dS = |d\vec{S}| = \sqrt{f_x^2 + f_y^2 + 1} \ dx \ dy. \) The conversion factor \( \sqrt{\cdot} \) between \( dS \) and \( dA \) relates area on \( S \) to area of projection in \( xy \)-plane.)

• Example: flux of \( \vec{F} = z \hat{k} \) through \( S \) portion of paraboloid \( z = x^2 + y^2 \) above unit disk, oriented with normal pointing up (and into the paraboloid): geometrically flux should be \( > 0 \) (asked using flashcards). We have \( \hat{n} \ dS = (-2x, -2y, 1) \ dx \ dy, \) and

\[
\iint_S \vec{F} \cdot d\vec{S} = \iint_S z \ dx \ dy = \iint_S (x^2 + y^2) \ dx \ dy = \int_0^{2\pi} \int_0^1 r^2 r \ dr \ d\theta = \pi / 2.
\]

**Parametric surfaces.** If we can describe \( S \) by parametric equations \( x = x(u, v), \ y = y(u, v), \ z = z(u, v) \) (i.e. \( \vec{r} = \vec{r}(u, v) \)), then we can set up flux integrals using variables \( u, v \). To find \( d\vec{S}, \)
consider a small portion of surface corresponding to changes $\Delta u$ and $\Delta v$ in parameters, it’s a parallelogram with sides $\vec{r}(u + \Delta u, v) - \vec{r}(u, v) \approx (\partial \vec{r}/\partial u) \Delta u$ and $(\partial \vec{r}/\partial v) \Delta v$, so

$$\Delta \vec{S} = \pm \left( \frac{\partial \vec{r}}{\partial u} \Delta u \right) \times \left( \frac{\partial \vec{r}}{\partial v} \Delta v \right), \quad \text{and} \quad d\vec{S} = \pm \left( \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du dv.$$  

(This generalizes all formulas previously seen; but won’t be needed on exam).

**Implicit surfaces:** If we have an implicitly defined surface $g(x, y, z) = 0$, then we have a (non-unit) normal vector $\mathbf{N} = \nabla g$. (similarly for a slanted plane, from equation $ax + by + cz = d$ we get $\mathbf{N} = (a, b, c)$).

Unit normal $\hat{\mathbf{n}} = \pm \mathbf{N}/|\mathbf{N}|$; surface element $\Delta S = \hat{\mathbf{n}} dS = \hat{\mathbf{n}} dA$, and $\hat{\mathbf{n}} dS = |\mathbf{N}| \hat{\mathbf{n}} (\mathbf{N} \cdot \nabla) dx dy = \pm \frac{|\mathbf{N}|}{\mathbf{N} \cdot \nabla} \hat{\mathbf{n}} dA$, I forgot the absolute value).

Note: if $S$ is vertical then the denominator is zero, can’t project to $xy$-plane any more (but one could project e.g. to the $xz$-plane).

**Example:** if $S$ is a graph, $g(x, y, z) = z - f(x, y) = 0$, then $\mathbf{N} = \langle g_x, g_y, g_z \rangle = \langle -f_x, -f_y, 1 \rangle$, $\mathbf{N} \cdot \hat{\mathbf{e}} = 1$, so we recover the formula $dS = (-f_x, -f_y, 1) dx dy$ seen before.

**Divergence theorem.** (”Gauss-Green theorem”) – 3D analogue of Green theorem for flux.

If $S$ is a closed surface bounding a region $D$, with normal pointing outwards, and $\vec{F}$ vector field defined and differentiable over all of $D$, then

$$\int_S \vec{F} \cdot d\vec{S} = \int\int_D \text{div} \vec{F} \, dV,$$  \quad \text{where} \quad \text{div} (P \hat{\mathbf{i}} + Q \hat{\mathbf{j}} + R \hat{\mathbf{k}}) = P_x + Q_y + R_z.$$

**Example:** flux of $\vec{F} = z \hat{\mathbf{k}}$ out of sphere of radius $a$ (seen Thursday): $\text{div} \vec{F} = 0 + 0 + 1 = 1$, so $\int_S \vec{F} \cdot d\vec{S} = 3 \text{vol}(D) = 4\pi a^3/3$.

**Physical interpretation** (mentioned very quickly and verbally only): $\text{div} \vec{F}$ = source rate = flux generated per unit volume. So the divergence theorem says: the flux outwards through $S$ (net amount leaving $D$ per unit time) is equal to the total amount of sources in $D$.  