Lecture 7: Continuation and Exam Review

Hyperbolic Sine and Cosine

Hyperbolic sine (pronounced “sinsh”):

\[ \sinh(x) = \frac{e^x - e^{-x}}{2} \]

Hyperbolic cosine (pronounced “cosh”):

\[ \cosh(x) = \frac{e^x + e^{-x}}{2} \]

\[ \frac{d}{dx} \sinh(x) = \frac{d}{dx} \left( \frac{e^x - e^{-x}}{2} \right) = \frac{e^x - (-e^{-x})}{2} = \cosh(x) \]

Likewise,

\[ \frac{d}{dx} \cosh(x) = \sinh(x) \]

(Note that this is different from \( \frac{d}{dx} \cos(x) \).)

Important identity:

\[ \cosh^2(x) - \sinh^2(x) = 1 \]

Proof:

\[
\begin{align*}
\cosh^2(x) - \sinh^2(x) &= \left( \frac{e^x + e^{-x}}{2} \right)^2 - \left( \frac{e^x - e^{-x}}{2} \right)^2 \\
&= \frac{1}{4} (e^{2x} + 2e^x e^{-x} + e^{-2x}) - \frac{1}{4} (e^{2x} - 2 + e^{-2x}) \\
&= \frac{1}{4} (2 + 2) = 1
\end{align*}
\]

Why are these functions called “hyperbolic”?

Let \( u = \cosh(x) \) and \( v = \sinh(x) \), then

\[ u^2 - v^2 = 1 \]

which is the equation of a hyperbola.

Regular trig functions are “circular” functions. If \( u = \cos(x) \) and \( v = \sin(x) \), then

\[ u^2 + v^2 = 1 \]

which is the equation of a circle.
Exam 1 Review

General Differentiation Formulas

\[
(u + v)' = u' + v' \\
(cu)' = cu' \\
(uv)' = u'v + uv' \quad \text{(product rule)} \\
\left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2} \quad \text{(quotient rule)} \\
\frac{d}{dx} f(u(x)) = f'(u(x)) \cdot u'(x) \quad \text{(chain rule)}
\]

You can remember the quotient rule by rewriting

\[
\left( \frac{u}{v} \right)' = (uv^{-1})'
\]
and applying the product rule and chain rule.

Implicit differentiation

Let’s say you want to find \( y' \) from an equation like

\[ y^3 + 3xy^2 = 8 \]

Instead of solving for \( y \) and then taking its derivative, just take \( \frac{d}{dx} \) of the whole thing. In this example,

\[
3y^2y' + 6xyy' + 3y^2 = 0 \\
(3y^2 + 6xy)y' = -3y^2 \\
y' = \frac{-3y^2}{3y^2 + 6xy}
\]

Note that this formula for \( y' \) involves both \( x \) and \( y \). Implicit differentiation can be very useful for taking the derivatives of inverse functions.

For instance,

\[ y = \sin^{-1} x \Rightarrow \sin y = x \]

Implicit differentiation yields

\[(\cos y)y' = 1\]

and

\[ y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}} \]
Specific differentiation formulas

You will be responsible for knowing formulas for the derivatives and how to deduce these formulas from previous information: \( x^n, \sin^{-1} x, \tan^{-1} x, \sin x, \cos x, \tan x, \sec x, e^x, \ln x \).

For example, let’s calculate \( \frac{d}{dx} \sec x \):

\[
\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = \frac{-(-\sin x)}{\cos^2 x} = \tan x \sec x
\]

You may be asked to find \( \frac{d}{dx} \sin x \) or \( \frac{d}{dx} \cos x \), using the following information:

\[
\lim_{h \to 0} \frac{\sin(h)}{h} = 1
\]
\[
\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0
\]

Remember the definition of the derivative:

\[
\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

Tying up a loose end

How to find \( \frac{d}{dx} x^r \), where \( r \) is a real (but not necessarily rational) number? All we have done so far is the case of rational numbers, using implicit differentiation. We can do this two ways:

1st method: base \( e \)

\[
x = e^{\ln x}
\]
\[
x^r = (e^{\ln x})^r = e^{r \ln x}
\]
\[
\frac{d}{dx} x^r = \frac{d}{dx} e^{r \ln x} = e^{r \ln x} \frac{d}{dx} (r \ln x) = e^{r \ln x} \frac{r}{x}
\]
\[
\frac{d}{dx} x^r = x^r \left( \frac{r}{x} \right) = rx^{r-1}
\]

2nd method: logarithmic differentiation

\[
(ln f)' = \frac{f'}{f}
\]
\[
f = x^r
\]
\[
\ln f = \ln x
\]
\[
(ln f)' = \frac{r}{x}
\]
\[
f' = f (ln f)' = x^r \left( \frac{r}{x} \right) = rx^{r-1}
\]
Finally, in the first lecture I promised you that you’d learn to differentiate anything—even something as complicated as

\[
\frac{d}{dx} e^{x \tan^{-1} x}
\]

So let’s do it!

\[
\frac{d}{dx} e^{uv} = e^{uv} \frac{d}{dx} (uv) = e^{uv} (u'v + uv')
\]

Substituting,

\[
\frac{d}{dx} e^{x \tan^{-1} x} = e^{x \tan^{-1} x} \left( \tan^{-1} x + x \left( \frac{1}{1 + x^2} \right) \right)
\]