Lecture 5
Implicit Differentiation and Inverses

Implicit Differentiation

Example 1.  \( \frac{d}{dx} (x^a) = ax^{a-1} \).
We proved this by an explicit computation for \( a = 0, 1, 2, ... \). From this, we also got the formula for \( a = -1, -2, ... \) Let us try to extend this formula to cover rational numbers, as well:

\[
a = \frac{m}{n}; \quad y = x^n \quad \text{where } m \text{ and } n \text{ are integers.}
\]

We want to compute \( \frac{dy}{dx} \). We can say \( y^n = x^m \) so \( ny^{n-1} \frac{dy}{dx} = mx^{m-1} \). Solve for \( \frac{dy}{dx} \):

\[
\frac{dy}{dx} = \frac{m}{n} x^{m-1} y^{n-1}
\]

We know that \( y = x^{\left(\frac{m}{n}\right)} \) is a function of \( x \).

\[
\frac{dy}{dx} = \frac{m}{n} \left( x^{m-1} \right) \left( y^n \right)^{n-1} \\
= \frac{m}{n} \left( x^{m-1} \right) \left( \left( x^m \right)^n \right)^{n-1} \\
= \frac{m}{n} x^{m(n-1)/n} \\
= \frac{m}{n} x^{(m-1) - \frac{m(n-1)}{n}} \\
= \frac{m}{n} x^{n(m-1) - m(n-1)} \\
= \frac{m}{n} x^{nm - nm + m} \\
= \frac{m}{n} x^{m - n} \\
= \frac{m}{n} x^{\frac{m}{n}} - 1
\]

So, \( \frac{dy}{dx} = \frac{m}{n} x^{\frac{m}{n}} - 1 \)

This is the same answer as we were hoping to get!

Example 2.  Equation of a circle with a radius of 1: \( x^2 + y^2 = 1 \) which we can write as \( y^2 = 1 - x^2 \). So \( y = \pm \sqrt{1 - x^2} \). Let us look at the positive case:

\[
y = \sqrt{1 - x^2} = (1 - x^2)^{\frac{1}{2}} \\
\frac{dy}{dx} = \left( \frac{1}{2} \right) (1 - x^2)^{-\frac{1}{2}} (-2x) = \frac{-x}{\sqrt{1 - x^2}} = \frac{-x}{y}
\]
Now, let’s do the same thing, using implicit differentiation.

\[ x^2 + y^2 = 1 \]

\[
\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (1) = 0
\]

\[
\frac{d}{dx} (x^2) + \frac{d}{dx} (y^2) = 0
\]

Applying chain rule in the second term,

\[ 2x + 2y \frac{dy}{dx} = 0 \]

\[ 2y \frac{dy}{dx} = -2x \]

\[ \frac{dy}{dx} = -\frac{x}{y} \]

Same answer!

**Example 3.** \( y^3 + xy^2 + 1 = 0 \). In this case, it’s not easy to solve for \( y \) as a function of \( x \). Instead, we use implicit differentiation to find \( \frac{dy}{dx} \).

\[
3y^2 \frac{dy}{dx} + y^2 + 2xy \frac{dy}{dx} = 0
\]

We can now solve for \( \frac{dy}{dx} \) in terms of \( y \) and \( x \).

\[
\frac{dy}{dx} (3y^2 + 2xy) = -y^2
\]

\[
\frac{dy}{dx} = -\frac{y^2}{3y^2 + 2xy}
\]

**Inverse Functions**

If \( y = f(x) \) and \( g(y) = x \), we call \( g \) the inverse function of \( f \), \( f^{-1} \):

\[ x = g(y) = f^{-1}(y) \]

Now, let us use implicit differentiation to find the derivative of the inverse function.

\[ y = f(x) \]

\[ f^{-1}(y) = x \]

\[
\frac{d}{dx} (f^{-1}(y)) = \frac{d}{dx} (x) = 1
\]

By the chain rule:

\[
\frac{d}{dy} (f^{-1}(y)) \frac{dy}{dx} = 1
\]

and

\[
\frac{d}{dy} (f^{-1}(y)) = \frac{1}{\frac{dy}{dx}}
\]
So, implicit differentiation makes it possible to find the derivative of the inverse function.

**Example.** \( y = \arctan(x) \)

\[
\begin{align*}
\tan y &= x \\
\frac{d}{dx} [\tan(y)] &= \frac{dx}{dx} = 1 \\
\frac{d}{dy} [\tan(y)] \frac{dy}{dx} &= 1 \\
\left( \frac{1}{\cos^2(y)} \right) \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \cos^2(y) = \cos^2(\arctan(x))
\end{align*}
\]

This form is messy. Let us use some geometry to simplify it.

![Diagram](image)

**Figure 1:** Triangle with angles and lengths corresponding to those in the example illustrating differentiation using the inverse function \( \arctan \)

In this triangle, \( \tan(y) = x \) so

\( \arctan(x) = y \)

The Pythagorean theorem tells us the length of the hypotenuse:

\( h = \sqrt{1 + x^2} \)

From this, we can find

\[ \cos(y) = \frac{1}{\sqrt{1 + x^2}} \]

From this, we get

\[ \cos^2(y) = \left( \frac{1}{\sqrt{1 + x^2}} \right)^2 = \frac{1}{1 + x^2} \]
So,

\[
\frac{dy}{dx} = \frac{1}{1 + x^2}
\]

In other words,

\[
\frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2}
\]

**Graphing an Inverse Function.**

Suppose \( y = f(x) \) and \( g(y) = f^{-1}(y) = x \). To graph \( g \) and \( f \) together we need to write \( g \) as a function of the variable \( x \). If \( g(x) = y \), then \( x = f(y) \), and what we have done is to trade the variables \( x \) and \( y \). This is illustrated in Fig. 2

\[
\begin{align*}
\quad f^{-1}(f(x)) &= x & f^{-1} \circ f(x) &= x \\
\quad f(f^{-1}(x)) &= x & f \circ f^{-1}(x) &= x
\end{align*}
\]

![Figure 2: You can think about \( f^{-1} \) as the graph of \( f \) reflected about the line \( y = x \)](image)