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Approximation of Random Fields in High Dimension

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Outline

1. Main objects and setup
   - Tensor product-type random fields
   - Average case information complexity

2. Exact asymptotic expression for the information complexity
   - Auxiliary probabilistic construction
   - Main result

3. Proof

4. Some references
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Definition of tensor product-type random fields

- for any $d = 1, 2, \ldots$: separable zero-mean random function
  $\{X^{(d)}(t)\}$, $t \in [0, 1]^d$ with the covariance function

  $$K^{(d)}(s, t) = \prod_{\ell=1}^{d} K_{\ell}(s_{\ell}, t_{\ell})$$

  for all $s_{\ell}, t_{\ell} \in [0, 1]$, $s = (s_1, \ldots, s_d)$, $t = (t_1, \ldots, t_d)$.

- family of tensor product-type random fields

  $$X = \left\{ X^{(d)}(t), t \in [0, 1]^d \right\}, \quad d = 1, 2, \ldots$$

  $d \to \infty$. 

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Some examples of tensor product-type random fields

- **Wiener-Chentsov random field:**

\[
\mathcal{K}_{W(d)}(s, t) = \prod_{\ell=1}^{d} \min\{s_\ell, t_\ell\}.
\]

- **\(d\)-variate completely tucked Brownian sheet (Hoefding, Blum, Kiefer and Rosenblatt process):**

\[
\mathcal{K}_{B(d)}(s, t) = \prod_{\ell=1}^{d} (\min\{s_\ell, t_\ell\} - s_\ell t_\ell).
\]

\(s_\ell, t_\ell \in [0, 1], \ s = (s_1, \ldots, s_d), \ t = (t_1, \ldots, t_d).\)
Tensor product-type random fields

Karhunen-Loève expansion

$$\Lambda := \sum_{i=1}^{\infty} \lambda_i^2 < \infty.$$  

$$\lambda_i^2$$ and $$\varphi_i(\cdot): \lambda_i^2 \varphi_i(t) = \int_0^1 K(s, t) \varphi_i(s) ds, \ t \in [0, 1].$$

The covariance operator of $$X^{(d)}$$ has the eigenvalues:

$$\lambda_k^2 := \lambda_{k_1}^2 \lambda_{k_2}^2 \ldots \lambda_{k_d}^2, \ k = (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d.$$

Karhunen-Loève expansion: with $$\varphi_k(t) = \varphi_{k_1}(t_1) \ldots \varphi_{k_d}(t_d)$$

$$X^{(d)}(t) = \sum_{k \in \mathbb{N}^d} \xi_k \lambda_k \varphi_k(t), \ t = (t_1, \ldots, t_d) \in [0, 1]^d, \quad (1.1)$$

where $$\{\xi_k\}$$ is an array of non-correlated $$\mathcal{N}(0, 1)$$ r.v.’s.
Let $X := X^{(d)}$ and $X_n$ be the partial sum of the Karhunen-Loève expansion (1.1) corresponding to $n$ maximal eigenvalues.

**The average case information complexity:**

$$n(\varepsilon, d) := \min\{n : \mathbb{E}\|X - X_n\|^2 \leq \varepsilon^2 \Lambda^d\},$$

where

$$\Lambda^d = \left(\sum_{i=1}^{\infty} \lambda_i^2\right)^d = \sum_{k \in \mathbb{N}^d} \lambda_k^2 = \mathbb{E}\|X\|^2.$$

**Aim:** the asymptotic behavior of $n(\varepsilon, d)$, as $d \to \infty$. 
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Auxiliary sequence of i.i.d. r.v.'s

\begin{align*}
\text{i.i.d. } \{ U_\ell \}: & \quad \mathbb{P}(U_\ell = - \log \lambda_i) = \frac{\lambda_i^2}{\Lambda}, \ i = 1, 2, \ldots. \\
(3dM): & \quad \sum_{i=1}^{\infty} | \log \lambda_i |^3 \lambda_i^2 < \infty, \text{ then } \mathbb{E}|U_\ell|^3 < \infty. \\
M & := \mathbb{E}U_\ell = - \sum_{i=1}^{\infty} \log \lambda_i \frac{\lambda_i^2}{\Lambda}, \\
\sigma^2 & := \text{Var } U_\ell = \sum_{i=1}^{\infty} | \log \lambda_i |^2 \frac{\lambda_i^2}{\Lambda} - M^2. \\
\alpha^3 & := \mathbb{E}(U_\ell - M)^3 = - \sum_{i=1}^{\infty} (\log \lambda_i)^3 \frac{\lambda_i^2}{\Lambda} - 3M \sigma^2 - M^3.
\end{align*}
Explosion coefficient

\[ \mathcal{E} := \Lambda e^{2M}. \]

**Lemma**

\[ \mathcal{E} > 1. \]
\[ \mathcal{E} = 1 \iff \sigma = 0. \]

Proof: by concavity of the logarithmic function.
Recall: $\mathcal{E} = \Lambda e^{2M} > 1$.

**Theorem**

Let $\{\lambda_i\}_{i \geq 1}$ satisfy $(3dM)$. Then for every $\varepsilon \in (0, 1)$ it holds

$$n(\varepsilon, d) = K \phi(q_\varepsilon/\sigma) \mathcal{E}^d e^{2q_\varepsilon \sqrt{d}} d^{-1/2} (1 + o(1)), \quad d \to \infty,$$

where $\phi(\cdot)$ is the standard normal density, the constant $K$ is known and depends on whether $U_\ell$ has a lattice distribution or not, and the quantile $q = q_\varepsilon$ is chosen from the equation

$$1 - \Phi(q/\sigma) = \varepsilon^2.$$

Curse of dimensionality: $\log n(\varepsilon, d) = d \log \mathcal{E} (1 + o(1)), \quad d \to \infty$. 
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First step: sum of units

\[ \zeta := \zeta(\varepsilon, d) : \text{the maximal positive number s.t.:} \]

\[ \sum_{k \in \mathbb{N}^d : \lambda_k < \zeta} \lambda_k^2 = \mathbb{E} \| X - X_n \|^2 \leq \varepsilon^2 \Lambda^d. \]

\[ A = A(\varepsilon, d) := \left\{ k \in \mathbb{N}^d : \lambda_k \geq \zeta \right\} = \left\{ k \in \mathbb{N}^d : \prod_{\ell=1}^{d} \lambda_{k_{\ell}} \geq \zeta \right\}. \]
First step: sum of units. (Cont.)

Remember: \( \lambda_i^2 = \Lambda \mathbb{P}(U_\ell = -\log \lambda_i), \ i = 1, 2, \ldots \)

\( \lambda_k > 0 \) for any \( k \in A \):

\[
n(\varepsilon, d) = \text{card}(A) = \sum_{k \in A} \frac{\lambda_k^2}{\lambda^2} = \sum_{k \in \mathbb{N}^d : \sum U_\ell \leq -\log \zeta} \exp \left\{ -2 \sum_{\ell=1}^{d} \log \lambda_{k_\ell} \right\} \Lambda^d \prod_{\ell=1}^{d} \mathbb{P}(U_\ell = -\log \lambda_{k_\ell})
\]

\[
= \Lambda^d \mathbb{E} \exp \left\{ 2 \sum_{\ell=1}^{d} U_\ell \right\} \mathbb{I}\{\sum_{\ell=1}^{d} U_\ell \leq -\log \zeta\}.
\]
Second step: quantile convergence

For any $d \in \mathbb{N}$, $z \in \mathbb{R}^1$ and fixed $\varepsilon \in (0, 1)$:

\[
\sum_{k \in \mathbb{N}^d: \lambda_k < z} \lambda_k^2 = \Lambda^d \mathbb{P} \left( \sum_{l=1}^{d} U_l > -\log z \right)
\]

\[
= \Lambda^d \mathbb{P} \left( Z_d > -\frac{\log z + dM}{\sigma \sqrt{d}} \right) = \Lambda^d \mathbb{P} (Z_d > \theta_z) \leq \varepsilon^2 \Lambda^d,
\]

where $\theta_z = -\frac{\log z + dM}{\sigma \sqrt{d}}$, and $Z_d := \frac{\sum_{\ell=1}^{d} U_{\ell} - dM}{\sigma \sqrt{d}}$.

Then $\theta = \theta(\varepsilon, d)$ is the $(1 - \varepsilon^2)$-quantile of the d.f. of $Z_d$.

By the CLT: $\theta(\varepsilon, d) \to q_\varepsilon / \sigma$, $d \to \infty$, where $q = q_\varepsilon$ is the quantile of the normal d.f.
Thus, for the information complexity we obtained:

\[
n(\varepsilon, d) = \mathcal{E}^d \mathbb{E} \exp\{2\sigma \sqrt{d} Z_d\} \mathbb{I}\{Z_d \leq \theta\} = \mathcal{E}^d \exp\{2\sigma \sqrt{d} \theta\} \int_{-\infty}^{\theta} \exp\{2\sigma \sqrt{d} (z - \theta)\} \, dF_d(z),
\]

where \( F_d(z) = \mathbb{P}(Z_d < z) \), \( \mathcal{E} := \Lambda e^{2M} \),

\[
Z_d := \frac{\sum_{\ell=1}^{d} U_\ell - dM}{\sigma \sqrt{d}},
\]

\[
\theta = \theta(\varepsilon, d) := -\frac{\log \zeta + dM}{\sigma \sqrt{d}}.
\]
Third step: application of the Edgeworth-type expansions

- The rest of the proof:
  - integrate by parts
  - apply the Cramér-Esseen Theorems:

if the distribution of $U_\ell$ is not lattice, it holds (uniformly in $z$):

$$F_d(z) = \Phi(z) - \frac{e^{-z^2/2}}{\sqrt{2\pi}} \frac{\alpha^3(z^2 - 1)}{6\sigma^3\sqrt{d}} + o \left( \frac{1}{\sqrt{d}} \right)$$

and with the additional periodic term otherwise:

$$e^{-z^2/2}hS \left( \frac{(z\sigma\sqrt{d} - da)/h}{\sigma\sqrt{2\pi d}} \right),$$

where $S(x) := [x] - x + \frac{1}{2}$, $a$ is a shift and $h$ is the maximal span of the distribution of $U_\ell - \bar{M}$. 
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