Symmetries and Regular Patterns In Natural and Man-Made Objects

"Symmetry is a complexity-reducing concept [...] seek it everywhere."

Alan J. Perlis

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Structure Discovery in 3D PCD
Point Cloud Data (PCD) Pose Particular Challenges

- PCD = “point cloud data”
  - unorganized collection of points sampled from the surface (or interior) of an object, with noise added
  - typical output of a 3-D scanning process
- no connectivity information or manifold or mesh structure ⇒ hard to use geometric methods directly
- no regular sampling
  ⇒ hard to use signal processing tools
Shared Structure Across Distributed Data Sets

- Data sets of interest may be distributed over a network
- May be massive
- May have different owners

- How to decide when data sets should be, or can be, fused, compared, etc?
Computational Symmetry: Detecting Self-Similarity

I. Symmetry Extraction and Symmetrization
II. Distributed Congruence Discovery
III. Repeated Pattern Detection

Global Structure Discovery

Continuous
Discrete

Geometry
Multiscale Analysis
I. Symmetry Extraction and Symmetrization

[Mitra, G., Pauly, Siggraph ’06, Mitra, G., Pauly, Siggraph ’07]
Partial/Approximate Symmetry Detection

**Given:**
Object/shape (represented as point cloud, mesh, ... )

**Goal:**
Identify and extract similar (symmetric) patches of possibly different sizes, across different resolutions
An Example: Reflective Symmetry
Reflective Symmetry: A Pair Votes
Reflective Symmetry : Voting Continues
Reflective Symmetry: Voting Continues
Reflective Symmetry : Largest Cluster

- Height of cluster $\rightarrow$ size of patch
- Spread of cluster $\rightarrow$ approximation level
A Typical Density Plot

height of cluster → extent of approximate symmetry

spread of cluster → deviation from exact symmetry
Pipeline

Note that even if a shape is perfectly symmetric, its sampling need not be so.
Pruning: Local Signatures

- Local signature $\rightarrow$ invariant under transforms
- Signatures disagree $\rightarrow$ points don’t correspond

Use $(\kappa_1, \kappa_2)$ for curvature based pruning
Reflection: Normal-Based Pruning

[Diagram of a butterfly with various labeled components and processes such as input model, sample set, signatures, transformations, density, and surface patches, with arrows indicating the flow between these components.]
Point Pair Pruning

- All pairs
- Curvature based
- Curvature + normal based
Transformations

- Reflection → point-pairs
- Rigid transform → more information

Robust estimation of principal curvature frames
[Cohen-Steiner et al. `03]
Mean-Shift Clustering

Kernel:
- **Type** → radially symmetric hat function
- **Radius**
Verification

- Clustering gives a good guess of the dominant symmetries
- Suggested symmetries need to be verified against the data
- Locally refine transforms using ICP algorithm
  [Besl and McKay `92]
Random Sampling

- Height of clusters related to symmetric region size
- Larger regions likely to be detected earlier
- Output-sensitive ...
Compression: Chambord
Compression: Chambord
Sidney Opera
Approximate Symmetry: Dragon

detected symmetries
correction field
Extrinsic vs. Intrinsic Symmetries

**Extrinsic Symmetry**
- Invariance under translation, rotation, reflection and scaling (Isometries of the ambient space)
- Break under isometric deformations of the shape

**Intrinsic Symmetry**
- Invariance of geodesic distances under self-mappings. For a homeomorphism $T: O \to O$
  \[ g(p, q) = g(T(p), T(q)) \forall p, q \in O \]
- Persist under isometric deformations
- Introduced by Raviv et al. in NRTL 2007
Global Intrinsic Symmetries

- **Signature space**
  - For each point \( p \) define its signature \( s(p) \) [Rustamov, SGP 2007]
    \[
    s(p) = \left( \frac{\phi_1(p)}{\sqrt{\lambda_1}}, \frac{\phi_2(p)}{\sqrt{\lambda_2}}, ..., \frac{\phi_i(p)}{\sqrt{\lambda_i}}, ... \right)
    \]
  - \( \phi_i(p) \) is the value of the \( i \)-th eigenfunction of the Laplace-Beltrami operator at \( p \)
  - Invariant under isometric deformations

- **Main Observation:** Intrinsic symmetries of the object become extrinsic symmetries in the signature space.

1. \( \phi = \phi \circ T \): **positive** eigenfunction
2. \( \phi = -\phi \circ T \): **negative** eigenfunction
3. \( \lambda \) is a repeated eigenvalue
Global Intrinsic Symmetries
Partial Intrinsic Symmetries

- One part of an object is isometrically mapped to another part
- Use heat kernel

\[ k_t(x, y) = \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y). \]

- Is the amount of heat transferred from \( y \) to \( x \) in \( t \) time.
- \( k_t(x, \cdot) \) is a bump function with scale \( t \)

\( k_t(x, x) \), a function of \( t \) at each \( x \) — as a shape signature

[Ovsjanikov, Sun, G., in prep. 2008]
Extrinsic Symmetrization

Goal: Symmetrize 3D geometry

Approach: Minimally deform the model in the spatial domain by optimizing the distribution in transformation space
Cluster Enhancement and Contraction

shape after cluster contraction

cluster merging achieves global symmetry

cluster merging achieves global symmetry
Symmetrization Demo

Cluster Contraction
Key Points and Issues

- Capturing partial/approximate/intrinsic symmetries of 3D shapes can be done efficiently via a voting mechanism.
- Only transforms supported by the data are searched and larger symmetries are found with less work.
II. Distributed Congruence Discovery

[Pauly, Giesen, Mitra, G., SGP 2006]
Probabilistic Fingerprints

- Probabilistic fingerprint
- Shapes are never directly compared
- Compact
- Compare
- Independent

Partial similarity
Insight

Partial matching $\rightarrow$ difficult problem
Total matching $\rightarrow$ easy problem

Reduce partial matching $\rightarrow$
many small total matching problems

Results in few false positives $\rightarrow$
quick to verify and discard

From document similarity to shape similarity: shingles and min-hashing
Input Shapes
Sample Points
Shingles: Overlapping Patches
Shingles: Overlapping Patches
Bag of Patches: Ordering Discarded

... but with sufficient overlaps, can be recovered – DNA whole genome shotgun sequencing
Fingerprint Pipeline
Pipeline: Uniform Sampling

- Uniform spacing → use [Turk`92]
- Sample spacing ≈ δ
Pipeline: Shingle Generation

- Shingles: overlapping, unordered patches
- Shingle radius: $\rho$
- $\rho \gg \delta$
Pipeline: Signatures

- Stable signatures wrt. sampling (continuity)
- Invariant to rigid transforms
  - Spin-images [Johnson, Hebert 1999]
- Shape → unordered high-dimensional point set with rigid transform factored out
Pipeline: Resemblance

- Similarity/resemblance
- Defined wrt. signatures
- Compare two bags of points in a high-d space
- No alignment required
- Still, brute force evaluation impractical

Jaccard similarity measure

$$r(S_1, S_2) = \frac{|S_1 \cap S_2|}{|S_1 \cup S_2|}$$
How to Compare Point Sets

- Compare two point sets → no need to align
  \[ r(S_1, S_2) = \frac{|S_1 \cap S_2|}{|S_1 \cup S_2|} \]
- But, we don’t have red and blue points together
Reduce Sample Size

- Randomly sample red points
- Randomly sample blue points

We need consistent sampling

\{ independently

still need to solve for correspondences
Min-Hashing I: Using Random ‘Experts’

Each of $m$ random ‘experts’

- Has an ordering of space-boxes
- Selects the point that lies in lowest ordered box

$$\min\{\pi(I)\}$$
Min-Hashing II

Each of m random ‘experts’

- Has an ordering of space-box
- Selects the point that lies in lowest ordered box

\[ \min \{ \pi(I) \} \]
Pipeline: Min-Hashing

Feature selection by random experts

- ‘Features’ only useful for correspondence
- Need not have any visual or semantic importance
- Reduces set comparison to element-wise comparison
Data Reduction

quantization

min hashing

Shingles → Signatures → Descriptors → Fingerprint

100k → 100k → 100k → 1k

set size remains constant

set reduction
Applications

- Resemblance between partial scans
Applications

- Adaptive feature selection for stitching
Applications

- Multiple scans
  - greedy alignment using priority queue
  - fingerprint matching determines score
- advanced alignment method for verification
- merging fingerprints requires no re-computation
Applications

• Shape distributions
Key Points and Issues

• Resemblance defined as set operation on signature sets → quantization is crucial
• Random experts effectively extract consistent set of features → requiring no explicit correspondences
• Fingerprints do not preserve spatial relation of shingles → false positives are possible
• Few parameters that are easy to tune
III. Repeated Pattern Detection

[Pauly, Mitra, Wallner. G., and Pottmann, Siggraph ’08]
Structure Discovery

- Discover regular structures in 3D data, without prior knowledge of either the pattern involved, or the repeating element.

- Algorithm has three stages:
  - Transformation analysis
  - Model estimation
  - Aggregation

Challenges: joint discrete and continuous optimization, presence of clutter and outliers
Algorithm Overview

Input Model → Transform Analysis → Transform Clusters

Structure Discovery

Aggregation

Regular Structures

Transform Generators

A. → B. → C.
Algorithm Overview

[Diagram showing the algorithm overview with steps from input model to transform domains through structure discovery and aggregation.]
Repetitive Structures

1D structures

2D structures

Regular structures:
rotation + translation + scaling → any commutative combinations in the form of 1D, 2D grid structures
Similarity Sets

Compare all pairs of small patches, using local shape descriptors

Based on shape descriptors alone
Pruned, after validation w. geometric alignment
Transform Analysis

- Regularity in the spatial domain is enhanced in the transform domain
Density Plots in Transform Space
Model Estimation: Where is the Grid?
Grid Fitting with Clutter and Outliers

\[ \tilde{g}_1, \tilde{g}_2, \{\alpha_{ij}\}, \{\beta_k\} = \arg\min_{\tilde{g}_1, \tilde{g}_2, \{\alpha_{ij}\}, \{\beta_k\}} E \]

\[ E = \gamma(E_{X \rightarrow C} + E_{C \rightarrow X}) + (1 - \gamma)(E_\alpha + E_\beta) \]

\[ E_{X \rightarrow C} = \sum_i \sum_j \alpha_{ij}^2 \left\| \tilde{x}_{ij} - \tilde{c}(i, j) \right\|^2 \]

\[ E_{C \rightarrow X} = \sum_{k=1}^{|C|} \beta_k^2 \left\| \tilde{c}_k - \tilde{x}(k) \right\|^2 \]

\[ E_\alpha = \sum_i \sum_j (1 - \alpha_{ij})^2 \quad E_\beta = \sum_k (1 - \beta_k^2)^2 \]

\[ X = \text{grid} \quad C = \text{transform cluster} \]
Aggregation

Once the basic repeated pattern is determined, we simultaneously (re-)optimize the pattern generators and the repeating geometric element it represents, going back to the original 3D data.

We inteleave

- region growing
- re-optimization of the generating transforms of the pattern by performing simultaneous registrations on the original geometry.
We optimize a generating transform \( T \) represented by 4x4 matrix \( H \), by trying to improve the alignment of all patches put into correspondence by \( T \), using standard ICP techniques.

\[
H_+ \approx H + \epsilon D \cdot H,
\]

\[
D = \begin{pmatrix}
\delta & -d_3 & d_2 & \bar{d}_1 \\
d_3 & \delta & -d_1 & \bar{d}_2 \\
-d_2 & d_1 & \delta & \bar{d}_3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
T_+(\bar{x}) \approx T(\bar{x}) + \epsilon (\bar{d} \times T(\bar{x}) + \delta T(\bar{x}) + \bar{d})
\]

\[
T_+^k \approx (H + \epsilon \bar{D} \cdot \bar{H})^k \rightarrow \bar{H}_+^k \approx \bar{H}^k + \epsilon f_k(\bar{H}, \bar{D}) + \epsilon^2(\ldots), \quad \text{with}
\]

\[
f_k(\bar{H}, \bar{D}) = \bar{D} \cdot \bar{H}^k + \bar{H} \cdot \bar{D} \cdot \bar{H}^{k-1} + \cdots + \bar{H}^{k-1} \cdot \bar{D} \cdot \bar{H}
\]

\[
Q_{ij} := \sum_l \left( [(T_+^k(\bar{x}_l) - \bar{y}_l) \cdot \bar{n}_l]^2 + \mu [T_+^k(\bar{x}_l) - \bar{y}_l]^2 \right)
\]

\[
F(\epsilon \bar{D}) = \sum_{i,j} Q_{ij}
\]
Scanned Building Facade

Output:
- Golden: 7x3 2D grid
- Blue: 5x3 2D grid
Back to Chambord (30-100K Sample Points)
Amphitheater
Amphitheater

Output: 3 grids + associated patches
Robustness to Missing Data
Nautilus: Similarity Transform

Input: 72 registered laser scans
Nautilus: Similarity Transform

Output: Detected structure + growth
Key Points and Issues

- Patterns in 3D data map into patterns in the space of locally aligning transforms
- Grid fitting w. weights as optimization variables allows for missing data and outliers
- The full geometry is exploited in detecting the optimal repeating element and pattern generator(s)
- Related to non-local smoothing in images
- Understand the Kolmogorov complexity of data
- Deal w. deformed patterns
Geometric Structure Extraction as a Paradigm for Data Analysis

- All of science and engineering is becoming data rich
  - massive data coming from sensors
  - massive data coming from simulations
- Such data from physical processes is often in the form of unorganized point clouds
- Machine learning is fundamentally based on fitting functions to data (regression, classification)
- An alternative approach can be comparing data to itself, or to other data of the same type

Physical Laws \( \equiv \) Symmetries
Challenge: From 3-D to Any-D

Many interesting PCD data sets from sensors and simulations are higher dimensional.

Can these techniques be applied to such settings (especially, low-d data sets in high-d ambient space)?

I. How do we estimate good local descriptors for high-dimensional data?
II. What if the data is sparse?
III. Are there “structure-preserving” low-d projections and embeddings?

Can we handle dynamic data changes, streaming data sets, etc?
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