2. Incidence Geometry

Incidence geometry

- An **incidence geometry** (G,c) of **rank** k is a graph G with a proper vertrex coloring c, where k colors are used.
- Sometimes we denote the geometry by (G,~,T,c). Here c:V(G) ! T is the coloring and |T| = k is the number of colors, also known as the **rank** of G. The relation ~ is called the **incidence**.
- T is the set of **types**. Note that only objects of different types may be incident.

Morphisms or representations

- Given two incidence geometries (G,~,c,T) and (G',~,c',T') a pair (f,g) of mappings
- f: G ! G' and
- g: T ! T' is called a **morphism** of geometries (or **representation**) if the following is true:
  - for any v 2 V(G) c'(f(v)) = g(c(v)).
  - for any u,v 2 V(G): if u ~ v then either f(u) = f(v) or f(u) ~ f(v).

Special morphisms

- Some morphisms have nice properties and deserve special attention.
- We call a representation **dimension-preserving** if
  - for any u,v 2 V(G): if u ~ v then f(u) ~ f(v).
- We call a representation **faithful** or **strong** if
  - for any u,v 2 V(G): u ~ v if and only if f(u) ~ f(v).
- A faithful representation in which both f and g are injective is called **realization**.
- A morphism is an **isomorphism** if both f and g are bijections and the inverse pair $(f^{-1},g^{-1´})$ is a morphism too.
- The image of a representation is geometry. The image of a realization is isomorphic to the original.
- For stirng geometries we seek representations and realizations in sets. Vertices are mapped to the elements (or singletons) of S and the faces to subsets of S. The incidence $u_i \sim u_j$, i < j is represented by inclusions $S(u_i)$ µ $S(u_j)$.

Automorphisms

- There are two types of automorphisms in a geometry (G,~,c,T). Aut_0 G contains type-preserving automorphisms. (g = id). Aut G contains all (extended) automorphisms.
- In the case of string geometries we want the linear order on T to be respected (or reversed). In the case of extended automorphisms we speak of dualities that map faces of rank r to rank n-r.

Examples

- 1. Each incidence structure is a rank 2 geometry. (Actualy, look at its Levi graph.)
- 2. Each 3 dimensional polyhedron is a rank 3 geometry. There are three types of objects: vertices, edges and faces with obvious geometric incidence.
- 3. Each (abstract) simplicial complex is an incidence geometry. Incidence is defined by inclusion of simplices.
- 4. Any complete multipartite graph is a geometry. Take for instance $K_{2,2,2}$, $K_{2,2,2,2}$, $K_{2,2,\ldots,2}$. The vertex coloring defining the geometry in each case is obvious.

Pasini Geometry

- Pasini defines incidence geometry (that we call **Pasini geometry**) in a more restrictive way.
  - For $k=1$, the graph must contain at least two vertices: $|V(G)|>1$.
  - For $k>1$:
    - G has to be connected,
    - For each $x \in V(G)$ the $(k-1)$-colored graph (G_x,c), called **residuum**, induced on the neigbors of x is a Pasini geometry of rank $(k-1)$.

String geometries

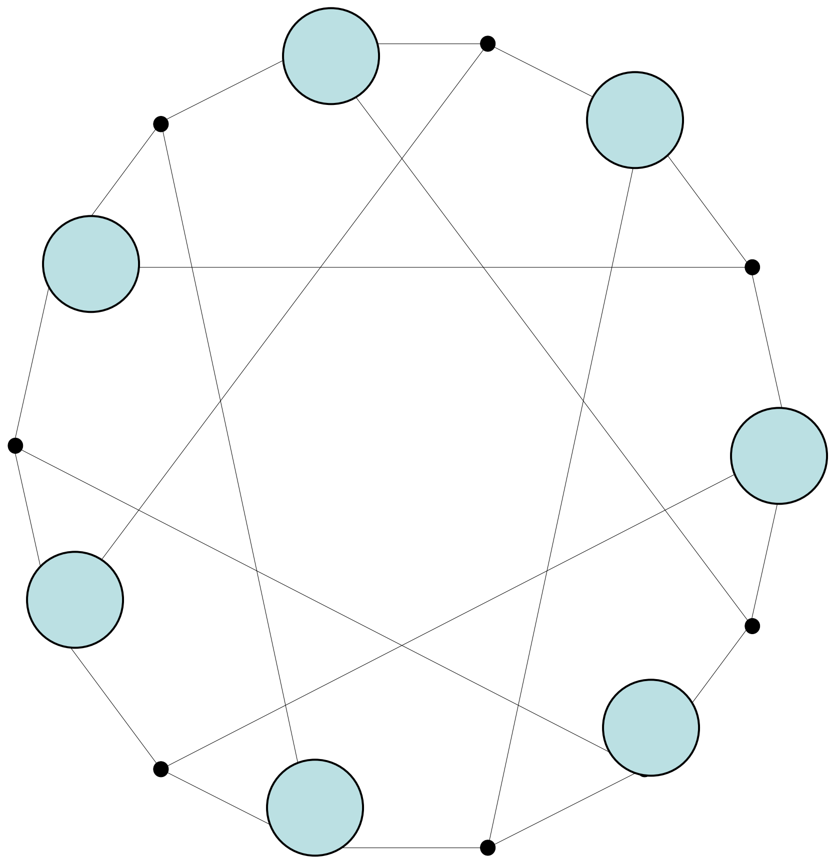
- A geometry G over the set of types T = {-1,0,1, ..., n} is called a **string geometry** if the following (1-2) is true (the elements of G are called **faces,** faces of type 0 are called **vertices** (or **points**), faces of type 1 are called **edges** (or **lines**), faces of type n-1 are called **facets**.). It is called **pure string geometry** if (1-3) is true.

1. There are exactly two **improper faces** u_{-1} 2 V(G) of type -1 and one element u_n 2 V(G) of type n (both incident with every other face). The rest are called **proper faces**.
2. If u_i, u_j, u_k are elements of respective types $i < j < k$ and $u_i \sim u_j$, $u_j \sim u_k$, then $u_i \sim u_k$.
3. Every collection of mutually incident faces U can be extended to a sequence of (n+2) mutually incident faces. (In other words: all chambers have rank n+2.)

Incidence geometries of rank 2

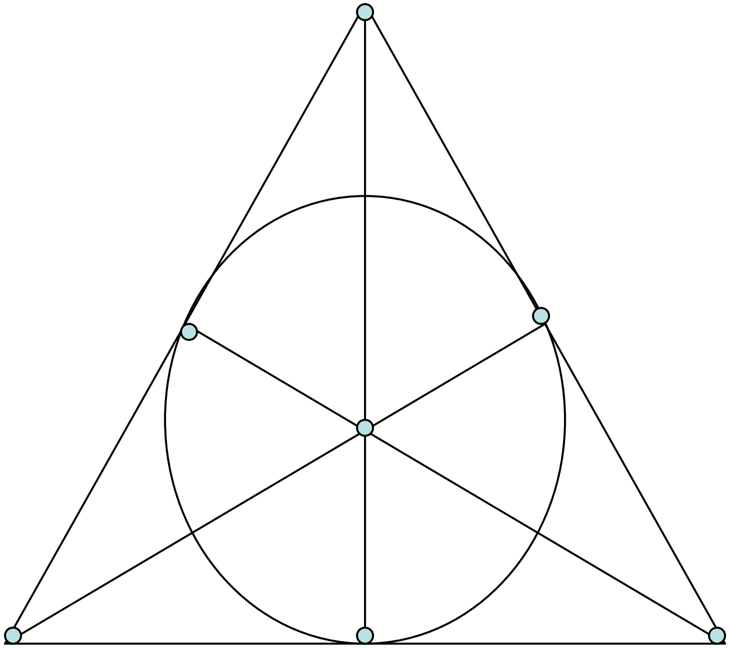
- Incidence geometries of rank 2 are simply bipartite graphs with a given black and white vertex coloring.
- Rank 2 Pasini geometries are in addition connected and the valence of each vertex is at least 2: $\delta(G) > 1$.

Example of Rank 2 Geometry



- Graph H on the left is known as the **Heawood graph**.
- H is connected
- H is trivalent: $\delta(H) = \Delta(H) = 3$.
- H is bipartite.
- H is a Pasini geometry.

Another View

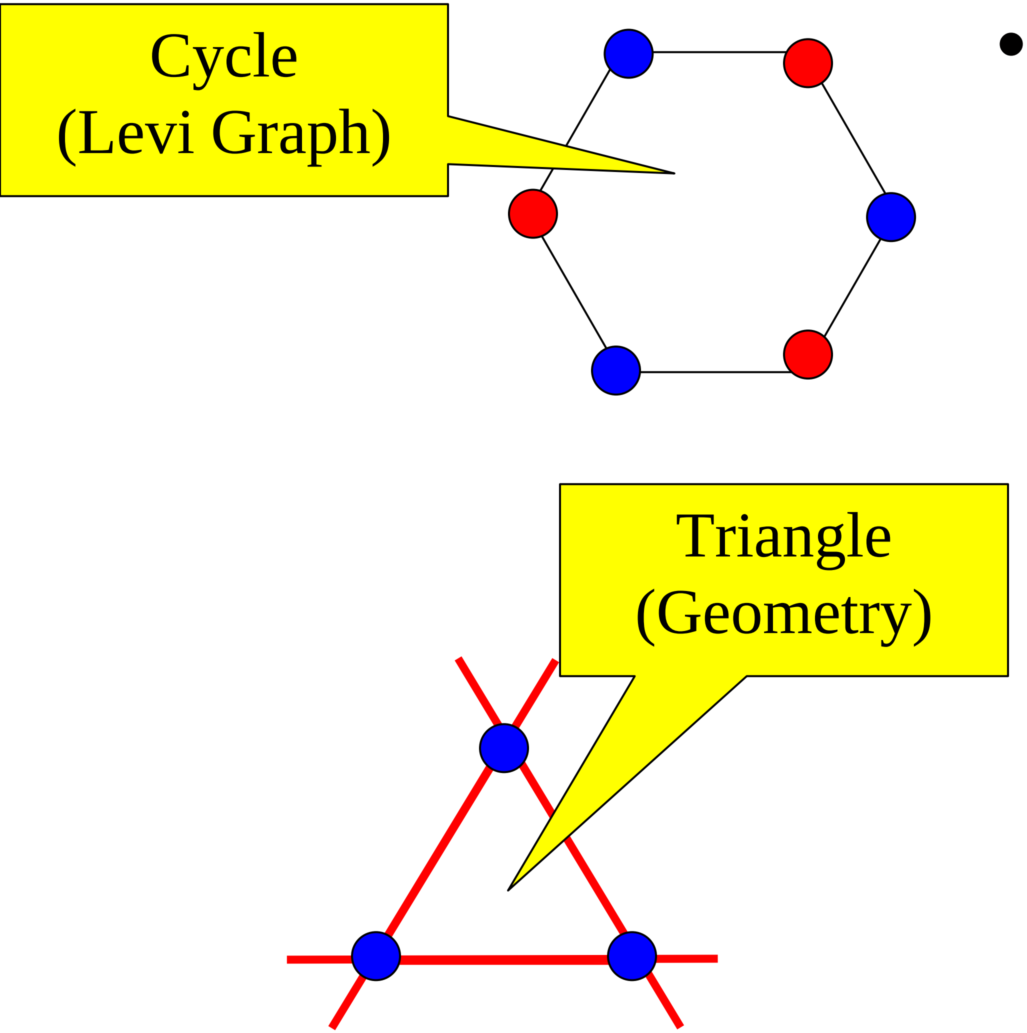


- The geometry of the Heawood graph H has another interpretation.
- Rank = 2. There are two types of objects in Euclidean plane, say, **points** and **curves.**
- There are 7 points, 7 curves, 3 points on a curve, 3 curves through a point.
- The corresponding Levi graph is H!

In other words ...

- The Heawood graph (with a given black and white coloring) is the same thing as the Fano plane (7_3), the smallest finite projective plane.
- Any incidence geometry can be interpeted in terms of abstract points, lines.
- If we want to distinguish the geometry (interpretation) from the associated graph we refer to the latter as the **Levi graph** of the corresponding geometry.

Simplest Rank 2 Pasini Geometries



- “Simplest” geometries of rank 2 in the sense of Pasini are even cycles. For instance the Levi graph C_6 corresponds to the triangle.

Rank 3

- Incidence geometries of rank 3 are exactly 3-colored graphs.
- Pasini geometries of rank 3 are much more restricted. Currently we are interested in those geometries whose residua are even cycles.
- Such geometries correspond to Eulerian surface triangulations with a given vertex 3-coloring.

Flag System as Geometries

- Any flag system Φ µ V £ E £ F defines a rank 3 geometry on X = V t E t F. There are three types of elements and two distinct elements of X are incicent if and only if they belong to the same flag of Φ.

Self-avoiding maps

- Recall that a map is **self-avoiding** if and only if neither the skeleton of the map nor the skeleton of its dual has a loop.

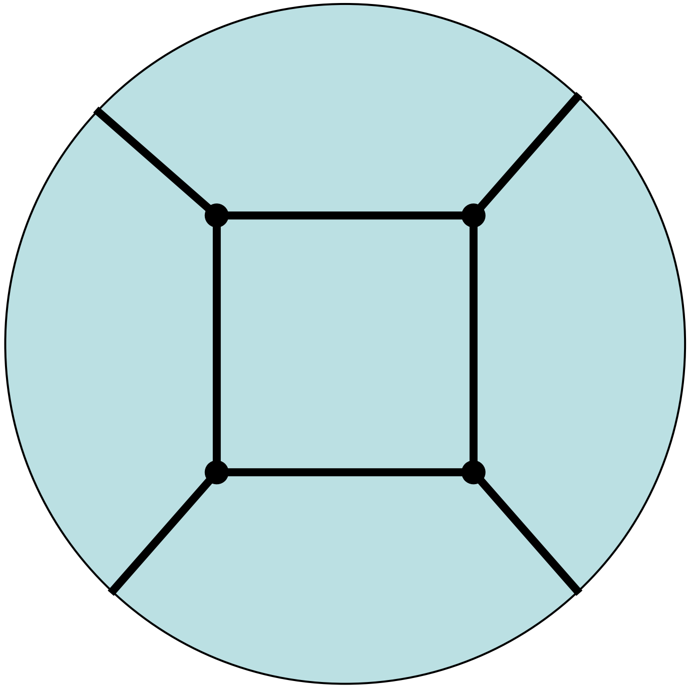
Self-avoiding maps as Geometries of rank 4

- Consider a generalized flag system Φ µ V £ E £ F £ P that defines a rank 4 geometry on X = V t E t F t P.
- There are four types of elements and two distinct elements of X are incident if and only if they belong to the same flag of Φ.
- We may take any self-avoiding map M and the four involutions τ_0,τ_1,τ_2 and τ_3 and define a geometry as above.

Exercises 2

- N1. Prove that the Petrie dual of a self-avoiding map is self-avoiding.
- N2. Prove that any operation Du,Tr,Me,Su1, ... of a self-avoiding map is self-avoiding.
- N3. Prove that BS of any map is self-avoiding.
- N4. Show that any self-avoiding map may be considered as a geometry of rank 4 (add the fourth involution).

Homework 2



- H1 Describe the rank 4 geometry of the projective planar map on the left.