2. Incidence Geometry
Incidence geometry

- An **incidence geometry** $(G,c)$ of **rank** $k$ is a graph $G$ with a proper vertex coloring $c$, where $k$ colors are used.
- Sometimes we denote the geometry by $(G,\sim,T,c)$. Here $c:V(G) \rightarrow T$ is the coloring and $|T| = k$ is the number of colors, also known as the **rank** of $G$. The relation $\sim$ is called the **incidence**.
- $T$ is the set of **types**. Note that only objects of different types may be incident.
Morphisms or representations

- Given two incidence geometries \((G,\sim,c,T)\) and \((G',\sim,c',T')\) a pair \((f,g)\) of mappings \(f: G \rightarrow G'\) and \(g: T \rightarrow T'\) is called a morphism of geometries (or representation) if the following is true:
  - for any \(v \in V(G)\) \(c'(f(v)) = g(c(v))\).
  - for any \(u,v \in V(G)\): if \(u \sim v\) then either \(f(u) = f(v)\) or \(f(u) \sim f(v)\).
Special morphisms

- Some morphisms have nice properties and deserve special attention.
- We call a representation **dimension-preserving** if
  - for any $u,v \in V(G)$: if $u \sim v$ then $f(u) \sim f(v)$.
- We call a representation **faithful** or **strong** if
  - for any $u,v \in V(G)$: $u \sim v$ if and only if $f(u) \sim f(v)$.
- A faithful representation in which both $f$ and $g$ are injective is called **realization**.
- A morphism is an **isomorphism** if both $f$ and $g$ are bijections and the inverse pair $(f^{-1}, g^{-1})$ is a morphism too.
- The image of a representation is geometry. The image of a realization is isomorphic to the original.
- For string geometries we seek representations and realizations in sets. Vertices are mapped to the elements (or singletons) of $S$ and the faces to subsets of $S$. The incidence $u_i \sim u_j$, $i < j$ is represented by inclusions $S(u_i) \subseteq S(u_j)$.
Automorphisms

- There are two types of automorphisms in a geometry \((G, \sim, c, T)\). \(\text{Aut}_0 G\) contains type-preserving automorphisms. \((g = \text{id})\). \(\text{Aut} \ G\) contains all (extended) automorphisms.
- In the case of string geometries we want the linear order on \(T\) to be respected (or reversed). In the case of extended automorphisms we speak of dualities that map faces of rank \(r\) to rank \(n-r\).
Examples

• 1. Each incidence structure is a rank 2 geometry. (Actually, look at its Levi graph.)

• 2. Each 3 dimensional polyhedron is a rank 3 geometry. There are three types of objects: vertices, edges and faces with obvious geometric incidence.

• 3. Each (abstract) simplicial complex is an incidence geometry. Incidence is defined by inclusion of simplices.

• 4. Any complete multipartite graph is a geometry. Take for instance $K_{2,2,2}$, $K_{2,2,2,2}$, $K_{2,2}$, ..., 2. The vertex coloring defining the geometry in each case is obvious.
Pasini Geometry

• Pasini defines incidence geometry (that we call Pasini geometry) in a more restrictive way.
  – For $k=1$, the graph must contain at least two vertices: $|V(G)| > 1$.
  – For $k>1$:
    • $G$ has to be connected,
    • For each $x \in V(G)$ the $(k-1)$-colored graph $(G_x,c)$, called residuum, induced on the neighbors of $x$ is a Pasini geometry of rank $(k-1)$.
String geometries

- A geometry $G$ over the set of types $T = \{-1,0,1, \ldots, n\}$ is called a **string geometry** if the following (1-2) is true (the elements of $G$ are called **faces**, faces of type 0 are called **vertices** (or **points**), faces of type 1 are called **edges** (or **lines**), faces of type $n-1$ are called **facets**). It is called **pure string geometry** if (1-3) is true.

1. There are exactly two **improper faces** $u_{-1} \in V(G)$ of type -1 and one element $u_n \in V(G)$ of type $n$ (both incident with every other face). The rest are called **proper faces**.

2. If $u_i, u_j, u_k$ are elements of respective types $i < j < k$ and $u_i \sim u_j, u_j \sim u_k$, then $u_i \sim u_k$.

3. Every collection of mutually incident faces $U$ can be extended to a sequence of $(n+2)$ mutually incident faces. (In other words: all chambers have rank $n+2$.)
Incidence geometries of rank 2

• Incidence geometries of rank 2 are simply bipartite graphs with a given black and white vertex coloring.

• Rank 2 Pasini geometries are in addition connected and the valence of each vertex is at least 2: $\delta(G) > 1$. 
Example of Rank 2 Geometry

- Graph H on the left is known as the **Heawood graph**.
- H is connected
- H is trivalent: $\delta(H) = \Delta(H) = 3$.
- H is bipartite.
- H is a Pasini geometry.
Another View

- The geometry of the Heawood graph $H$ has another interpretation.
- Rank = 2. There are two types of objects in Euclidean plane, say, points and curves.
- There are 7 points, 7 curves, 3 points on a curve, 3 curves through a point.
- The corresponding Levi graph is $H$!
In other words ...

- The Heawood graph (with a given black and white coloring) is the same thing as the Fano plane (7\_3), the smallest finite projective plane.
- Any incidence geometry can be interpreted in terms of abstract points, lines.
- If we want to distinguish the geometry (interpretation) from the associated graph we refer to the latter as the **Levi graph** of the corresponding geometry.
Simplest Rank 2 Pasini Geometries

• “Simplest” geometries of rank 2 in the sense of Pasini are even cycles. For instance the Levi graph $C_6$ corresponds to the triangle.
Rank 3

• Incidence geometries of rank 3 are exactly 3-colored graphs.
• Pasini geometries of rank 3 are much more restricted. Currently we are interested in those geometries whose residua are even cycles.
• Such geometries correspond to Eulerian surface triangulations with a given vertex 3-coloring.
Flag System as Geometries

• Any flag system $\Phi \mu V \subseteq E \subseteq F$ defines a rank 3 geometry on $X = V \cup E \cup F$. There are three types of elements and two distinct elements of $X$ are incident if and only if they belong to the same flag of $\Phi$. 
Self-avoiding maps

• Recall that a map is **self-avoiding** if and only if neither the skeleton of the map nor the skeleton of its dual has a loop.
Self-avoiding maps as Geometries of rank 4

• Consider a generalized flag system $\Phi \mu V \mathcal{E} E \mathcal{F} \mathcal{E} P$ that defines a rank 4 geometry on $X = V \mathcal{E} E \mathcal{F} \mathcal{E} P$.

• There are four types of elements and two distinct elements of $X$ are incident if and only if they belong to the same flag of $\Phi$.

• We may take any self-avoiding map $M$ and the four involutions $\tau_0, \tau_1, \tau_2$ and $\tau_3$ and define a geometry as above.
Exercises 2

• N1. Prove that the Petrie dual of a self-avoiding map is self-avoiding.

• N2. Prove that any operation $Du, Tr, Me, Su_1, \ldots$ of a self-avoiding map is self-avoiding.

• N3. Prove that BS of any map is self-avoiding.

• N4. Show that any self-avoiding map may be considered as a geometry of rank 4 (add the fourth involution).
Homework 2

• H1 Describe the rank 4 geometry of the projective planar map on the left.