Graph embeddings and symmetries

1. Vertex-transitive maps

J. Širáň

Open Univ. and Slovak Tech. U.
Which graphs can be embedded on some surface in such a way that the resulting map is vertex-transitive? Of course, a necessary condition is that the graph itself be vertex-transitive—but is this sufficient? What about other kinds of transitivity of a map, such as edge-, dart-, or face-transitivity?

Overview of the lecture:
1. Surfaces, embeddings and maps
2. Existence of vertex-transitive maps
3. Construction of oriented maps
4. Algebra of maps and symmetries
Which graphs can be embedded on some surface in such a way that the resulting map is vertex-transitive?
Which **graphs** can be **embedded** on some **surface** in such a way that the resulting **map** is **vertex-transitive**?

Of course, a necessary condition is that the graph itself be vertex-transitive – but is this sufficient?
Which graphs can be embedded on some surface in such a way that the resulting map is vertex-transitive?

Of course, a necessary condition is that the graph itself be vertex-transitive – but is this sufficient?

What about other kinds of transitivity of a map, such as edge-, dart-, or face-transitivity?
Which graphs can be embedded on some surface in such a way that the resulting map is vertex-transitive?

Of course, a necessary condition is that the graph itself be vertex-transitive – but is this sufficient?

What about other kinds of transitivity of a map, such as edge-, dart-, or face-transitivity?

Overview of the lecture:
Which graphs can be embedded on some surface in such a way that the resulting map is vertex-transitive?

Of course, a necessary condition is that the graph itself be vertex-transitive – but is this sufficient?

What about other kinds of transitivity of a map, such as edge-, dart-, or face-transitivity?

Overview of the lecture:

1. Surfaces, embeddings and maps
Introduction

Which graphs can be embedded on some surface in such a way that the resulting map is vertex-transitive?

Of course, a necessary condition is that the graph itself be vertex-transitive – but is this sufficient?

What about other kinds of transitivity of a map, such as edge-, dart-, or face-transitivity?

Overview of the lecture:

1. Surfaces, embeddings and maps
2. Existence of vertex-transitive maps
Which graphs can be embedded on some surface in such a way that the resulting map is vertex-transitive?

Of course, a necessary condition is that the graph itself be vertex-transitive – but is this sufficient?

What about other kinds of transitivity of a map, such as edge-, dart-, or face-transitivity?

Overview of the lecture:

1. Surfaces, embeddings and maps
2. Existence of vertex-transitive maps
3. Construction of oriented maps
Which graphs can be embedded on some surface in such a way that the resulting map is vertex-transitive?

Of course, a necessary condition is that the graph itself be vertex-transitive – but is this sufficient?

What about other kinds of transitivity of a map, such as edge-, dart-, or face-transitivity?

Overview of the lecture:

1. Surfaces, embeddings and maps
2. Existence of vertex-transitive maps
3. Construction of oriented maps
4. Algebra of maps and symmetries
Introduction

Examples
Examples

Vertex-transitive embeddings of $K_4$ and $Q_3$ on a sphere:
Examples

Vertex-transitive embeddings of $K_4$ and $Q_3$ on a sphere:
Examples

Vertex-transitive embeddings of $K_4$ and $Q_3$ on a sphere:

An example of a vertex-transitive embedding of $K_5$ on a torus:
Examples

Vertex-transitive embeddings of $K_4$ and $Q_3$ on a sphere:

An example of a vertex-transitive embedding of $K_5$ on a torus:
Examples

An example of a vertex-transitive embedding of $K_7$ on a torus:
Examples

An example of a vertex-transitive embedding of $K_7$ on a torus:
Examples

An example of a vertex-transitive embedding of $K_7$ on a torus:

Exercise. Find vertex-transitive embeddings of $K_6$ and $K_{3,3}$ on a torus.
Examples

The Petersen graph on the projective plane, with its dual – $K_6$: 
The Petersen graph on the projective plane, with its dual \(- K_6\):

![Petersen graph on the projective plane with its dual - K_6](image-url)
The **Petersen graph** on the **projective plane**, with its **dual** – $K_6$:

Back to our question: How do we tell if a given graph embeds vertex-transitively?
The **Petersen graph** on the **projective plane**, with its dual – $K_6$:

Back to our question: How do we tell if a given graph embeds vertex-transitively?
Surfaces, embeddings and maps

The basics

Where do we want to embed graphs?
On "anything" that locally "behaves" like a plane!

A surface is a connected Hausdorff space in which every point has a neighbourhood homeomorphic to an open disc.

Classification of compact surfaces:
orientable: $S_g$ – sphere with $g \geq 0$ handles
nonorientable: $N_h$ – sphere with $h$ crosscaps

Exercise. Do not try to prove the above classification theorem.
The basics

Where do we want to embed graphs?
Where do we want to embed graphs?

On “anything” that locally “behaves” like a plane!
Where do we want to embed graphs?

On “anything” that locally “behaves” like a plane!

A surface is a connected Hausdorff space in which every point has a neighbourhood homeomorphic to an open disc.
The basics

Where do we want to embed graphs?

On “anything” that locally “behaves” like a plane!

A surface is a connected Hausdorff space in which every point has a neighbourhood homeomorphic to an open disc.

Classification of compact surfaces:
The basics

Where do we want to embed graphs?

On “anything” that locally “behaves” like a plane!

A surface is a connected Hausdorff space in which every point has a neighbourhood homeomorphic to an open disc.

Classification of compact surfaces:

- orientable: $S_g$ – sphere with $g \geq 0$ handles
Where do we want to embed graphs?

On “anything” that locally “behaves” like a plane!

A surface is a connected Hausdorff space in which every point has a neighbourhood homeomorphic to an open disc.

Classification of compact surfaces:

- orientable: \( S_g \) – sphere with \( g \geq 0 \) handles
- nonorientable: \( N_h \) – sphere with \( h \) crosscaps
Where do we want to embed graphs?

On “anything” that locally “behaves” like a plane!

A surface is a connected Hausdorff space in which every point has a neighbourhood homeomorphic to an open disc.

Classification of compact surfaces:

- orientable: $S_g$ – sphere with $g \geq 0$ handles
- nonorientable: $N_h$ – sphere with $h$ crosscaps

Exercise. Do not try to prove the above classification theorem.
Examples of surfaces
Examples of surfaces

Torus:
Examples of surfaces

Torus:
Examples of surfaces

Torus:

Double-torus:
Examples of surfaces

Torus:

Double-torus:
Examples of identification polygons
Examples of identification polygons

- Sphere
- Projective plane
- Torus
- Klein bottle
Examples of identification polygons

- Sphere
- Projective plane
- Torus
- Klein bottle
Examples of identification polygons

- Sphere
- Projective plane
- Torus
- Klein bottle
Informally, an embedding of a graph on a surface is a "drawing" with "no edge crossings". The embedding is cellular if "cutting the surface along the graph" results in "pieces" that are all "equivalent" to discs. Formally, if a graph \( \Gamma \) is viewed as a one-dimensional complex (with the natural topology), then an embedding of \( \Gamma \) on a surface \( S \) is a continuous injection \( j: \Gamma \to S \). The embedding \( j \) is 2-cell or cellular if each component of \( S \setminus j(\Gamma) \) is homeomorphic to an open disc. In such a case the pair \( (S, j(\Gamma)) \) is a map; each component of \( S \setminus j(\Gamma) \) is a face. Temporary restriction: We will consider only orientable surfaces. By preassigning an orientation of the surface we will make our maps oriented.
Informally, an embedding of a graph on a surface is a “drawing” with “no edge crossings”.

Formally, if a graph $\Gamma$ is viewed as a one-dimensional complex (with the natural topology), then an embedding of $\Gamma$ on a surface $S$ is a continuous injection $j: \Gamma \to S$. The embedding $j$ is 2-cellular if each component of $S \setminus j(\Gamma)$ is homeomorphic to an open disc. In such a case the pair $(S, j(\Gamma))$ is a map; each component of $S \setminus j(\Gamma)$ is a face.

Temporary restriction: We will consider only orientable surfaces. By preassigning an orientation of the surface we will make our maps oriented.
Informally, an embedding of a graph on a surface is a “drawing” with “no edge crossings”. The embedding is **cellular** if “cutting the surface along the graph” results in “pieces” that are all “equivalent” to discs.
Informally, an embedding of a graph on a surface is a “drawing” with “no edge crossings”. The embedding is cellular if “cutting the surface along the graph” results in “pieces” that are all “equivalent” to discs.

Formally,
Informally, an embedding of a graph on a surface is a “drawing” with “no edge crossings”. The embedding is cellular if “cutting the surface along the graph” results in “pieces” that are all “equivalent” to discs.

Formally, if a graph $\Gamma$ is viewed as a one-dimensional complex (with the natural topology),
Informally, an embedding of a graph on a surface is a “drawing” with “no edge crossings”. The embedding is cellular if “cutting the surface along the graph” results in “pieces” that are all “equivalent” to discs.

Formally, if a graph $\Gamma$ is viewed as a one-dimensional complex (with the natural topology), then an embedding of $\Gamma$ on a surface $S$ is a continuous injection $j : \Gamma \to S$. 
Informally, an embedding of a graph on a surface is a “drawing” with “no edge crossings”. The embedding is cellular if “cutting the surface along the graph” results in “pieces” that are all “equivalent” to discs.

Formally, if a graph $\Gamma$ is viewed as a one-dimensional complex (with the natural topology), then an embedding of $\Gamma$ on a surface $S$ is a continuous injection $j : \Gamma \rightarrow S$.

The embedding $j$ is 2-cell or cellular if each component of $S \setminus j(\Gamma)$ is homeomorphic to an open disc.
Informally, an embedding of a graph on a surface is a “drawing” with “no edge crossings”. The embedding is cellular if “cutting the surface along the graph” results in “pieces” that are all “equivalent” to discs.

Formally, if a graph $\Gamma$ is viewed as a one-dimensional complex (with the natural topology), then an embedding of $\Gamma$ on a surface $S$ is a continuous injection $j : \Gamma \rightarrow S$.

The embedding $j$ is 2-cell or cellular if each component of $S \setminus j(\Gamma)$ is homeomorphic to an open disc. In such a case the pair $(S, j(\Gamma))$ is a map; each component of $S \setminus j(\Gamma)$ is a face.
Informally, an embedding of a graph on a surface is a “drawing” with “no edge crossings”. The embedding is cellular if “cutting the surface along the graph” results in “pieces” that are all “equivalent” to discs.

Formally, if a graph $\Gamma$ is viewed as a one-dimensional complex (with the natural topology), then an embedding of $\Gamma$ on a surface $S$ is a continuous injection $j : \Gamma \rightarrow S$.

The embedding $j$ is 2-cell or cellular if each component of $S \setminus j(\Gamma)$ is homeomorphic to an open disc. In such a case the pair $(S, j(\Gamma))$ is a map; each component of $S \setminus j(\Gamma)$ is a face.

Temporary restriction: We will consider only orientable surfaces.
Embeddings

Informally, an embedding of a graph on a surface is a “drawing” with “no edge crossings”. The embedding is cellular if “cutting the surface along the graph” results in “pieces” that are all “equivalent” to discs.

Formally, if a graph $\Gamma$ is viewed as a one-dimensional complex (with the natural topology), then an embedding of $\Gamma$ on a surface $S$ is a continuous injection $j : \Gamma \to S$.

The embedding $j$ is 2-cell or cellular if each component of $S \setminus j(\Gamma)$ is homeomorphic to an open disc. In such a case the pair $(S, j(\Gamma))$ is a map; each component of $S \setminus j(\Gamma)$ is a face.

Temporary restriction: We will consider only orientable surfaces. By preassigning an orientation of the surface we will make our maps oriented.
In an oriented map, faces inherit the chosen orientation of the surface. This leads to thinking of edges as being formed by pairs of oppositely directed darts. Faces of the map are then bounded by facial walks.
Facial walks

In an oriented map, faces inherit the chosen orientation of the surface.
Facial walks

In an oriented map, faces inherit the chosen orientation of the surface.

This leads to thinking of edges as being formed by pairs of oppositely directed darts.
Facial walks

In an oriented map, faces inherit the chosen orientation of the surface.

This leads to thinking of edges as being formed by pairs of oppositely directed darts.

Faces of the map are then bounded by facial walks:
In an oriented map, faces inherit the chosen orientation of the surface.

This leads to thinking of edges as being formed by pairs of oppositely directed darts.

Faces of the map are then bounded by facial walks:
Existence of vertex-transitive maps

An automorphism (or, a symmetry) of an oriented map $M$ is a permutation of its darts which preserves facial walks and incidence. Group $\text{Aut}(M)$.

Let $M$ be a vertex-transitive map, that is, $\text{Aut}(M)$ is transitive on vertices of $\Gamma$.

Note: $\text{Aut}(M) < \text{Aut}(\Gamma)$.

Then, the stabiliser $\text{Stab}(v)$ of any vertex $v$ in the group $\text{Aut}(M)$ is a cyclic (possibly, trivial) subgroup! Moreover, the action of $\text{Stab}(v)$ on the darts at $v$ (i.e., pointing out of $v$) is free, which means that no non-trivial automorphism in $\text{Stab}(v)$ can fix a dart.

Thus, if $\Gamma$ embeds vertex-transitively on an oriented surface, then $\text{Aut}(\Gamma)$ has a vertex-transitive subgroup with cyclic free vertex stabilisers.

The converse is true as well: [J. Š., T. Tucker, 2007] Theorem 1. A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup with free cyclic vertex stabilisers.
An automorphism (or, a symmetry) of an oriented map $M$ is a permutation of its darts which preserves facial walks and incidence.
An automorphism (or, a symmetry) of an oriented map $M$ is a permutation of its darts which preserves facial walks and incidence. Group $\text{Aut}(M)$.
Existence of vertex-transitive maps

An **automorphism** (or, a **symmetry**) of an oriented map $M$ is a permutation of its darts which preserves facial walks and incidence. Group $\text{Aut}(M)$.

Let $M$ be a vertex-transitive map, that is, $\text{Aut}(M)$ is transitive on vertices of $\Gamma$.

Note: $\text{Aut}(M) < \text{Aut}(\Gamma)$.

Then, the stabiliser $\text{Stab}(v)$ of any vertex $v$ in the group $\text{Aut}(M)$ is a cyclic (possibly, trivial) subgroup. Moreover, the action of $\text{Stab}(v)$ on the darts at $v$ (i.e., pointing out of $v$) is free, which means that no non-trivial automorphism in $\text{Stab}(v)$ can fix a dart.

Thus, if $\Gamma$ embeds vertex-transitively on an oriented surface, then $\text{Aut}(\Gamma)$ has a vertex-transitive subgroup with cyclic free vertex stabilisers.

The converse is true as well: [J. Š., T. Tucker, 2007, Theorem 1.]

A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup with free cyclic vertex stabilisers.
An automorphism (or, a symmetry) of an oriented map $M$ is a permutation of its darts which preserves facial walks and incidence. Group $\text{Aut}(M)$.

Let $M$ be a vertex-transitive map, that is, $\text{Aut}(M)$ is transitive on vertices of $\Gamma$. Note: $\text{Aut}(M) < \text{Aut}(\Gamma)$. 

Thus, if $\Gamma$ embeds vertex-transitively on an oriented surface, then $\text{Aut}(\Gamma)$ has a vertex-transitive subgroup with cyclic free vertex stabilisers. The converse is true as well: [J. Š., T. Tucker, 2007] Theorem 1. A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup with free cyclic vertex stabilisers.
An automorphism (or, a symmetry) of an oriented map $M$ is a permutation of its darts which preserves facial walks and incidence. Group $\text{Aut}(M)$.

Let $M$ be a vertex-transitive map, that is, $\text{Aut}(M)$ is transitive on vertices of $\Gamma$. Note: $\text{Aut}(M) < \text{Aut}(\Gamma)$. Then, the stabiliser $\text{Stab}(v)$ of any vertex $v$ in the group $\text{Aut}(M)$ is a cyclic (possibly, trivial) subgroup!
Existence of vertex-transitive maps

An automorphism (or, a symmetry) of an oriented map $M$ is a permutation of its darts which preserves facial walks and incidence. Group $\text{Aut}(M)$.

Let $M$ be a vertex-transitive map, that is, $\text{Aut}(M)$ is transitive on vertices of $\Gamma$. Note: $\text{Aut}(M) < \text{Aut}(\Gamma)$. Then, the stabiliser $\text{Stab}(v)$ of any vertex $v$ in the group $\text{Aut}(M)$ is a cyclic (possibly, trivial) subgroup! Moreover, the action of $\text{Stab}(v)$ on the darts at $v$ (i.e., pointing out of $v$) is free.
Existence of vertex-transitive maps

An automorphism (or, a symmetry) of an oriented map $M$ is a permutation of its darts which preserves facial walks and incidence. Group $\text{Aut}(M)$.

Let $M$ be a vertex-transitive map, that is, $\text{Aut}(M)$ is transitive on vertices of $\Gamma$. Note: $\text{Aut}(M) < \text{Aut}(\Gamma)$. Then, the stabiliser $\text{Stab}(v)$ of any vertex $v$ in the group $\text{Aut}(M)$ is a cyclic (possibly, trivial) subgroup! Moreover, the action of $\text{Stab}(v)$ on the darts at $v$ (i.e., pointing out of $v$) is free, which means that no non-trivial automorphism in $\text{Stab}(v)$ can fix a dart.

The converse is true as well: [J. Širán, T. Tucker, 2007] Theorem 1. A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup with free cyclic vertex stabilisers.
Existence of vertex-transitive maps

An automorphism (or, a symmetry) of an oriented map $M$ is a permutation of its darts which preserves facial walks and incidence. Group $\text{Aut}(M)$.

Let $M$ be a vertex-transitive map, that is, $\text{Aut}(M)$ is transitive on vertices of $\Gamma$. Note: $\text{Aut}(M) < \text{Aut}(\Gamma)$. Then, the stabiliser $\text{Stab}(v)$ of any vertex $v$ in the group $\text{Aut}(M)$ is a cyclic (possibly, trivial) subgroup! Moreover, the action of $\text{Stab}(v)$ on the darts at $v$ (i.e., pointing out of $v$) is free, which means that no non-trivial automorphism in $\text{Stab}(v)$ can fix a dart.

Thus, if $\Gamma$ embeds vertex-transitively on an oriented surface,
Existence of vertex-transitive maps

An automorphism (or, a symmetry) of an oriented map $M$ is a permutation of its darts which preserves facial walks and incidence. Group $\text{Aut}(M)$.

Let $M$ be a vertex-transitive map, that is, $\text{Aut}(M)$ is transitive on vertices of $\Gamma$. Note: $\text{Aut}(M) < \text{Aut}(\Gamma)$. Then, the stabiliser $\text{Stab}(v)$ of any vertex $v$ in the group $\text{Aut}(M)$ is a cyclic (possibly, trivial) subgroup! Moreover, the action of $\text{Stab}(v)$ on the darts at $v$ (i.e., pointing out of $v$) is free, which means that no non-trivial automorphism in $\text{Stab}(v)$ can fix a dart.

Thus, if $\Gamma$ embeds vertex-transitively on an oriented surface, then $\text{Aut}(\Gamma)$ has a vertex-transitive subgroup with cyclic free vertex stabilisers.

The converse is true as well: [J. Š., T. Tucker, 2007] Theorem 1. A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup with free cyclic vertex stabilisers.
An automorphism (or, a symmetry) of an oriented map \( M \) is a permutation of its darts which preserves facial walks and incidence. Group \( \text{Aut}(M) \).

Let \( M \) be a vertex-transitive map, that is, \( \text{Aut}(M) \) is transitive on vertices of \( \Gamma \). Note: \( \text{Aut}(M) < \text{Aut}(\Gamma) \). Then, the stabiliser \( \text{Stab}(v) \) of any vertex \( v \) in the group \( \text{Aut}(M) \) is a cyclic (possibly, trivial) subgroup! Moreover, the action of \( \text{Stab}(v) \) on the darts at \( v \) (i.e., pointing out of \( v \)) is free, which means that no non-trivial automorphism in \( \text{Stab}(v) \) can fix a dart.

Thus, if \( \Gamma \) embeds vertex-transitively on an oriented surface, then \( \text{Aut}(\Gamma) \) has a vertex-transitive subgroup with cyclic free vertex stabilisers.

The converse is true as well: [J. Š., T. Tucker, 2007]
Existence of vertex-transitive maps

An automorphism (or, a symmetry) of an oriented map \( M \) is a permutation of its darts which preserves facial walks and incidence. Group \( \text{Aut}(M) \).

Let \( M \) be a vertex-transitive map, that is, \( \text{Aut}(M) \) is transitive on vertices of \( \Gamma \). Note: \( \text{Aut}(M) < \text{Aut}(\Gamma) \). Then, the stabiliser \( \text{Stab}(v) \) of any vertex \( v \) in the group \( \text{Aut}(M) \) is a cyclic (possibly, trivial) subgroup! Moreover, the action of \( \text{Stab}(v) \) on the darts at \( v \) (i.e., pointing out of \( v \)) is free, which means that no non-trivial automorphism in \( \text{Stab}(v) \) can fix a dart.

Thus, if \( \Gamma \) embeds vertex-transitively on an oriented surface, then \( \text{Aut}(\Gamma) \) has a vertex-transitive subgroup with cyclic free vertex stabilisers.

The converse is true as well: [J. Ġ., T. Tucker, 2007]

**Theorem 1.** A connected graph \( \Gamma \) has an oriented vertex-transitive embedding if and only if \( \text{Aut}(\Gamma) \) contains a vertex-transitive subgroup with free cyclic vertex stabilisers.
Illustration

Decide if the graph below admits an oriented vertex-transitive embedding.

Solution:

The graph can be re-drawn as follows:
Decide if the graph below admits an oriented vertex-transitive embedding.
Decide if the graph below admits an oriented vertex-transitive embedding.
Decide if the graph below admits an oriented vertex-transitive embedding. **Solution:** The graph can be re-drawn as follows:
Illustration

Decide if the graph below admits an oriented vertex-transitive embedding. 

Solution: The graph can be re-drawn as follows:
So, we have the Petersen graph $P$, 
So, we have the Petersen graph $P$, with vertex set “encoded” by two-element subsets of $\{1, 2, 3, 4, 5\}$, two vertices being adjacent if the corresponding sets are disjoint.
So, we have the Petersen graph $P$, with vertex set “encoded” by two-element subsets of $\{1, 2, 3, 4, 5\}$, two vertices being adjacent if the corresponding sets are disjoint. It can be shown that $\text{Aut}(P) \cong S_5$. 
So, we have the Petersen graph $P$, with vertex set “encoded” by two-element subsets of $\{1, 2, 3, 4, 5\}$, two vertices being adjacent if the corresponding sets are disjoint. It can be shown that $\text{Aut}(P) \cong S_5$.

By Theorem 1, $P$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(P) \cong S_5$ contains a subgroup transitive on vertices such that the vertex stabilisers are either trivial or isomorphic to $Z_3$. But the first case is impossible since $P$ is not a Cayley graph. The second case is excluded either, as $S_5$ is known to contain no subgroup of order 30 at all. Therefore the Petersen graph has no orientably vertex-transitive embedding.

This raises the following three questions:

1. Is the analysis of subgroups of $\text{Aut}(\Gamma)$ unavoidable?
2. If a suitable subgroup of $\text{Aut}(\Gamma)$ exists, how to get an embedding?
3. What are the supporting surfaces of such embeddings?
So, we have the Petersen graph $P$, with vertex set “encoded” by two-element subsets of $\{1, 2, 3, 4, 5\}$, two vertices being adjacent if the corresponding sets are disjoint. It can be shown that $Aut(P) \cong S_5$.

By Theorem 1, $P$ has an oriented vertex-transitive embedding if and only if $Aut(P) \cong S_5$ contains a subgroup transitive on vertices such that the vertex stabilisers are either trivial or isomorphic to $Z_3$. But the first case is impossible since $P$ is not a Cayley graph.
Illustration

So, we have the Petersen graph $P$, with vertex set “encoded” by two-element subsets of $\{1, 2, 3, 4, 5\}$, two vertices being adjacent if the corresponding sets are disjoint. It can be shown that $\text{Aut}(P) \cong S_5$.

By Theorem 1, $P$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(P) \cong S_5$ contains a subgroup transitive on vertices such that the vertex stabilisers are either trivial or isomorphic to $Z_3$. But the first case is impossible since $P$ is not a Cayley graph. The second case is excluded either, as $S_5$ is known to contain no subgroup of order 30 at all.

Therefore the Petersen graph has no orientably vertex-transitive embedding. This raises the following three questions:

1. Is the analysis of subgroups of $\text{Aut}(\Gamma)$ unavoidable?
2. If a suitable subgroup of $\text{Aut}(\Gamma)$ exists, how to get an embedding?
3. What are the supporting surfaces of such embeddings?

J. Širáň Open Univ. and Slovak Tech. U.
So, we have the Petersen graph $P$, with vertex set “encoded” by two-element subsets of \{1, 2, 3, 4, 5\}, two vertices being adjacent if the corresponding sets are disjoint. It can be shown that $\text{Aut}(P) \cong S_5$.

By Theorem 1, $P$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(P) \cong S_5$ contains a subgroup transitive on vertices such that the vertex stabilisers are either trivial or isomorphic to $Z_3$. But the first case is impossible since $P$ is not a Cayley graph. The second case is excluded either, as $S_5$ is known to contain no subgroup of order 30 at all. Therefore the Petersen graph has no orientably vertex-transitive embedding.
So, we have the Petersen graph $P$, with vertex set “encoded” by two-element subsets of $\{1, 2, 3, 4, 5\}$, two vertices being adjacent if the corresponding sets are disjoint. It can be shown that $\text{Aut}(P) \cong S_5$.

By Theorem 1, $P$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(P) \cong S_5$ contains a subgroup transitive on vertices such that the vertex stabilisers are either trivial or isomorphic to $Z_3$. But the first case is impossible since $P$ is not a Cayley graph. The second case is excluded either, as $S_5$ is known to contain no subgroup of order 30 at all. Therefore the Petersen graph has no orientably vertex-transitive embedding.

This raises the following three questions:

1. Is the analysis of subgroups of $\text{Aut}(\Gamma)$ unavoidable?
2. If a suitable subgroup of $\text{Aut}(\Gamma)$ exists, how to get an embedding?
3. What are the supporting surfaces of such embeddings?
Illustration

So, we have the Petersen graph $P$, with vertex set “encoded” by two-element subsets of $\{1, 2, 3, 4, 5\}$, two vertices being adjacent if the corresponding sets are disjoint. It can be shown that $Aut(P) \cong S_5$.

By Theorem 1, $P$ has an oriented vertex-transitive embedding if and only if $Aut(P) \cong S_5$ contains a subgroup transitive on vertices such that the vertex stabilisers are either trivial or isomorphic to $Z_3$. But the first case is impossible since $P$ is not a Cayley graph. The second case is excluded either, as $S_5$ is known to contain no subgroup of order 30 at all. Therefore the Petersen graph has no orientably vertex-transitive embedding.

This raises the following three questions:

1. Is the analysis of subgroups of $Aut(\Gamma)$ unavoidable?
So, we have the Petersen graph $P$, with vertex set “encoded” by two-element subsets of $\{1, 2, 3, 4, 5\}$, two vertices being adjacent if the corresponding sets are disjoint. It can be shown that $\text{Aut}(P) \cong S_5$.

By Theorem 1, $P$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(P) \cong S_5$ contains a subgroup transitive on vertices such that the vertex stabilisers are either trivial or isomorphic to $Z_3$. But the first case is impossible since $P$ is not a Cayley graph. The second case is excluded either, as $S_5$ is known to contain no subgroup of order 30 at all. Therefore the Petersen graph has no orientably vertex-transitive embedding.

This raises the following three questions:

1. Is the analysis of subgroups of $\text{Aut}(\Gamma)$ unavoidable?
2. If a suitable subgroup of $\text{Aut}(\Gamma)$ exists, how to get an embedding?
So, we have the Petersen graph $P$, with vertex set “encoded” by two-element subsets of $\{1, 2, 3, 4, 5\}$, two vertices being adjacent if the corresponding sets are disjoint. It can be shown that $\text{Aut}(P) \cong S_5$.

By Theorem 1, $P$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(P) \cong S_5$ contains a subgroup transitive on vertices such that the vertex stabilisers are either trivial or isomorphic to $\mathbb{Z}_3$. But the first case is impossible since $P$ is not a Cayley graph. The second case is excluded either, as $S_5$ is known to contain no subgroup of order 30 at all. Therefore the Petersen graph has no orientably vertex-transitive embedding.

This raises the following three questions:

1. Is the analysis of subgroups of $\text{Aut}(\Gamma)$ unavoidable?
2. If a suitable subgroup of $\text{Aut}(\Gamma)$ exists, how to get an embedding?
3. What are the supporting surfaces of such embeddings?
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$.

For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$.

This defines an involutory permutation $\lambda: D \to D$.

If $M$ is an oriented map with dart set $D$, we define another permutation $\rho$ of $D$, called rotation of $M$, as follows:

For any dart $b \in D$ at a vertex $v$, the dart $\rho(b)$ is the cyclically next dart at $v$ in the chosen orientation of $M$.

Important: Cycles of $\rho \lambda$ correspond to (directed) facial walks.

Note: $\rho \lambda(a) = b$, $\rho \lambda(b) = c$, etc., so $(a, b, c, d)$ is indeed a cycle of $\rho \lambda$. 
Recall: Edges of $\Gamma$ are viewed as pairs of darts;
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$. For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$. This defines an involutory permutation $\lambda: D \rightarrow D$. If $M$ is an oriented map with dart set $D$, we define another permutation $\rho$ of $D$, called the rotation of $M$, as follows: For any dart $b \in D$ at a vertex $v$, the dart $\rho(b)$ is the cyclically next dart at $v$ in the chosen orientation of $M$. Important: Cycles of $\rho \lambda$ correspond to (directed) facial walks. Note: $\rho \lambda(a) = b$, $\rho \lambda(b) = c$, etc., so $(a, b, c, d)$ is indeed a cycle of $\rho \lambda$. 
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$. For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$. This defines an involutory permutation $\lambda : D \to D$. If $M$ is an oriented map with dart set $D$, we define another permutation $\rho$ of $D$, called rotation of $M$, as follows: For any dart $b \in D$ at a vertex $v$, the dart $\rho(b)$ is the cyclically next dart at $v$ in the chosen orientation of $M$. Important: Cycles of $\rho \lambda$ correspond to (directed) facial walks. Note: $\rho \lambda(a) = b$, $\rho \lambda(b) = c$, etc., so $(a, b, c, d)$ is indeed a cycle of $\rho \lambda$. 

J. Širáň Open Univ. and Slovak Tech. U. ()
Graph embeddings and symmetries 1. Vertex-transitive maps 14 / 20
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$. For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$. This defines an involutory permutation $\lambda : D \to D$. 

Important: Cycles of $\rho \lambda$ correspond to (directed) facial walks. 

Note: $\rho \lambda(a) = b$, $\rho \lambda(b) = c$, etc., so $(a, b, c, d)$ is indeed a cycle of $\rho \lambda$. 

Algebraic approach to oriented maps

Permutation representation of maps
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$. For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$. This defines an involutory permutation $\lambda : D \to D$.

If $M$ is an oriented map with dart set $D$, we define another permutation $\rho$ of $D$, called rotation of $M$, as follows: For any dart $b \in D$ at a vertex $v$, the dart $\rho(b)$ is the cyclically next dart at $v$ in the chosen orientation of $M$.

Important: Cycles of $\rho \lambda$ correspond to (directed) facial walks.

Note: $\rho \lambda(a) = b$, $\rho \lambda(b) = c$, etc., so $(a, b, c, d)$ is indeed a cycle of $\rho \lambda$. 
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$. For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$. This defines an involutory permutation $\lambda : D \rightarrow D$.

If $M$ is an oriented map with dart set $D$, we define another permutation $\rho$ of $D$, called rotation of $M$, as follows:
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$. For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$. This defines an involutory permutation $\lambda : D \to D$.

If $M$ is an oriented map with dart set $D$, we define another permutation $\rho$ of $D$, called rotation of $M$, as follows: For any dart $b \in D$ at a vertex $v$, the dart $\rho(b)$ is the cyclically next dart at $v$ in the chosen orientation of $M$. Important: Cycles of $\rho \lambda$ correspond to (directed) facial walks. Note: $\rho \lambda(a) = b$, $\rho \lambda(b) = c$, etc., so $(a, b, c, d)$ is indeed a cycle of $\rho \lambda$. 

J. Širáň Open Univ. and Slovak Tech. U. (Graph embeddings and symmetries 1. Vertex-transitive maps 14 / 20)
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$. For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$. This defines an involutory permutation $\lambda : D \to D$.

If $M$ is an oriented map with dart set $D$, we define another permutation $\rho$ of $D$, called rotation of $M$, as follows: For any dart $b \in D$ at a vertex $v$, the dart $\rho(b)$ is the cyclically next dart at $v$ in the chosen orientation of $M$.

Important: Cycles of $\rho \lambda$ correspond to (directed) facial walks.
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$. For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$. This defines an involutory permutation $\lambda : D \rightarrow D$.

If $M$ is an oriented map with dart set $D$, we define another permutation $\rho$ of $D$, called rotation of $M$, as follows: For any dart $b \in D$ at a vertex $v$, the dart $\rho(b)$ is the cyclically next dart at $v$ in the chosen orientation of $M$.

Important: Cycles of $\rho \lambda$ correspond to (directed) facial walks.

Note: $\rho \lambda(a) = b$, 
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$. For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$. This defines an involutory permutation $\lambda : D \rightarrow D$.

If $M$ is an oriented map with dart set $D$, we define another permutation $\rho$ of $D$, called rotation of $M$, as follows: For any dart $b \in D$ at a vertex $v$, the dart $\rho(b)$ is the cyclically next dart at $v$ in the chosen orientation of $M$.

**Important:** Cycles of $\rho \lambda$ correspond to (directed) facial walks.

Note: $\rho \lambda(a) = b$, $\rho \lambda(b) = c$. 
Recall: Edges of $\Gamma$ are viewed as pairs of darts; let $D$ be the dart set of $\Gamma$. For a dart $b$ let $\lambda(b)$ be the reverse dart to $b$. This defines an involutory permutation $\lambda : D \rightarrow D$.

If $M$ is an oriented map with dart set $D$, we define another permutation $\rho$ of $D$, called rotation of $M$, as follows: For any dart $b \in D$ at a vertex $v$, the dart $\rho(b)$ is the cyclically next dart at $v$ in the chosen orientation of $M$.

Important: Cycles of $\rho \lambda$ correspond to (directed) facial walks.

Note: $\rho \lambda(a) = b$, $\rho \lambda(b) = c$, etc., so $(a, b, c, d)$ is indeed a cycle of $\rho \lambda$. 
Illustration
Example:

\[ D = \{ a, a', b, b', c, c', d, d', e, e', f, f' \}, \]

\[ \lambda(x) = x', \quad x \in \{ a, \ldots, f \}, \]

\[ \lambda^2 = \text{id}; \]

\[ \rho = (a', e, c)(b', d', e')(c', f', f, d, a). \]

Construct the oriented map given by \((\lambda, \rho)\).
Example:

\[ D = \{a, a', b, b', c, c', d, d', e, e', f, f'\}, \]
\[ \lambda(x) = x', x \in \{a, \ldots, f\}, \lambda^2 = \text{id}; \]
\[ \rho = (a', e, b)(b', d', e', c)(c', f', f, d, a). \]

Construct the oriented map given by \((\lambda, \rho)\).
Example:

\[ D = \{ a, a', b, b', c, c', d, d', e, e', f, f' \} , \]
\[ \lambda(x) = x' , x \in \{ a, \ldots , f \} , \lambda^2 = \text{id} ; \]
\[ \rho = (a', e, b)(b', d', e', c)(c', f', f, d, a) . \]
Construct the oriented map given by \((\lambda , \rho)\).

**Solution.** Vertices and faces of the map correspond to orbits of \(\rho\) and \(\rho \lambda\) where
\[ \rho \lambda = (a, e, c, f', d, e', b, d')(a', c', b')(f) : \]
The correspondence theorem and automorphisms

Theorem 2. Let $\Gamma$ be a connected graph with dart set $D$ and dart-reversing involution $\lambda$. Let $\rho$ be any permutation of $D$ such that, for each vertex $v$, $\rho$ is cyclic when restricted to darts at $v$. Then there is an oriented map $M$ with the underlying graph $\Gamma$ such that the rotation of $M$ is $\rho$. The genus $g$ of the supporting surface of $M$ is given by Euler's formula $|\rho| - |\lambda| + |\rho\lambda| = 2 - 2g$ where bars denote the number of orbits.∗

Fixed points in $\lambda$ can be allowed and they give rise to semi-edges.

A permutation $A$ of $D$ is an automorphism of $M = M(\lambda, \rho)$ if $A\rho = \rho A$ and $A\lambda = \lambda A$. This means that $\text{Aut}(M)$ is the centraliser of the group $\langle \lambda, \rho \rangle$ in the full symmetric group $\text{Sym}(D)$ of all permutations of $D$. 

∗J. Širáň
Theorem 2. Let $\Gamma$ be a connected graph with dart set $D$ and dart-reversing involution $\lambda$. 

The genus $g$ of the supporting surface of $M$ is given by Euler's formula $|\rho| - |\lambda| + |\rho\lambda| = 2 - 2g$ where bars denote the number of orbits.

Fixed points in $\lambda$ can be allowed and they give rise to semi-edges.

A permutation $A$ of $D$ is an automorphism of $M = M(\lambda, \rho)$ if $A\rho = \rho A$ and $A\lambda = \lambda A$.

This means that $\text{Aut}(M)$ is the centraliser of the group $\langle \lambda, \rho \rangle$ in the full symmetric group $\text{Sym}(D)$ of all permutations of $D$. 
Theorem 2. Let $\Gamma$ be a connected graph with dart set $D$ and dart-reversing involution $\lambda$. Let $\rho$ be any permutation of $D$ such that, for each vertex $v$, $\rho$ is cyclic when restricted to darts at $v$.
The correspondence theorem and automorphisms

**Theorem 2.** Let $\Gamma$ be a connected graph with dart set $D$ and dart-reversing involution $\lambda$. Let $\rho$ be any permutation of $D$ such that, for each vertex $v$, $\rho$ is cyclic when restricted to darts at $v$. Then there is an oriented map $M$ with the underlying graph $\Gamma$ such that the rotation of $M$ is $\rho$. 

The genus $g$ of the supporting surface of $M$ is given by Euler's formula $|\rho| - |\lambda| + |\rho\lambda| = 2 - 2g$ where bars denote the number of orbits.

Fixed points in $\lambda$ can be allowed and they give rise to semi-edges.

A permutation $A$ of $D$ is an automorphism of $M = M(\lambda, \rho)$ if $A\rho = \rho A$ and $A\lambda = \lambda A$.

This means that $\text{Aut}(M)$ is the centraliser of the group $\langle \lambda, \rho \rangle$ in the full symmetric group $\text{Sym}(D)$ of all permutations of $D$. 
Theorem 2. Let $\Gamma$ be a connected graph with dart set $D$ and dart-reversing involution $\lambda$. Let $\rho$ be any permutation of $D$ such that, for each vertex $v$, $\rho$ is cyclic when restricted to darts at $v$. Then there is an oriented map $M$ with the underlying graph $\Gamma$ such that the rotation of $M$ is $\rho$.

The genus $g$ of the supporting surface of $M$ is given by Euler’s formula
Theorem 2. Let $\Gamma$ be a connected graph with dart set $D$ and dart-reversing involution $\lambda$. Let $\rho$ be any permutation of $D$ such that, for each vertex $v$, $\rho$ is cyclic when restricted to darts at $v$. Then there is an oriented map $M$ with the underlying graph $\Gamma$ such that the rotation of $M$ is $\rho$.

The genus $g$ of the supporting surface of $M$ is given by Euler’s formula

$$|\rho| - |\lambda| + |\rho\lambda| = 2 - 2g$$
The correspondence theorem and automorphisms

Theorem 2. Let $\Gamma$ be a connected graph with dart set $D$ and dart-reversing involution $\lambda$. Let $\rho$ be any permutation of $D$ such that, for each vertex $v$, $\rho$ is cyclic when restricted to darts at $v$. Then there is an oriented map $M$ with the underlying graph $\Gamma$ such that the rotation of $M$ is $\rho$.

The genus $g$ of the supporting surface of $M$ is given by Euler’s formula

$$|\rho| - |\lambda| + |\rho\lambda| = 2 - 2g$$

where bars denote the number of orbits.
The correspondence theorem and automorphisms

**Theorem 2.** Let $\Gamma$ be a connected graph with dart set $D$ and dart-reversing involution $\lambda$. Let $\rho$ be any permutation of $D$ such that, for each vertex $v$, $\rho$ is cyclic when restricted to darts at $v$. Then there is an oriented map $M$ with the underlying graph $\Gamma$ such that the rotation of $M$ is $\rho$.

The genus $g$ of the supporting surface of $M$ is given by Euler’s formula

$$|\rho| - |\lambda| + |\rho\lambda| = 2 - 2g$$

where bars denote the number of orbits.

* Fixed points in $\lambda$ can be allowed and they give rise to semi-edges.
The correspondence theorem and automorphisms

Theorem 2. Let $\Gamma$ be a connected graph with dart set $D$ and dart-reversing involution $\lambda$. Let $\rho$ be any permutation of $D$ such that, for each vertex $v$, $\rho$ is cyclic when restricted to darts at $v$. Then there is an oriented map $M$ with the underlying graph $\Gamma$ such that the rotation of $M$ is $\rho$.

The genus $g$ of the supporting surface of $M$ is given by Euler’s formula

$$|\rho| - |\lambda| + |\rho\lambda| = 2 - 2g$$

where bars denote the number of orbits.

* Fixed points in $\lambda$ can be allowed and they give rise to semi-edges.

A permutation $A$ of $D$ is an automorphism of $M = M(\lambda, \rho)$ if $A\rho = \rho A$ and $A\lambda = \lambda A$. 
Theorem 2. Let \( \Gamma \) be a connected graph with dart set \( D \) and dart-reversing involution \( \lambda \). Let \( \rho \) be any permutation of \( D \) such that, for each vertex \( v \), \( \rho \) is cyclic when restricted to darts at \( v \). Then there is an oriented map \( M \) with the underlying graph \( \Gamma \) such that the rotation of \( M \) is \( \rho \).

The genus \( g \) of the supporting surface of \( M \) is given by Euler’s formula

\[
|\rho| - |\lambda| + |\rho \lambda| = 2 - 2g
\]

where bars denote the number of orbits.

* Fixed points in \( \lambda \) can be allowed and they give rise to semi-edges.

A permutation \( A \) of \( D \) is an automorphism of \( M = M(\lambda, \rho) \) if \( A\rho = \rho A \) and \( A\lambda = \lambda A \).

This means that \( \text{Aut}(M) \) is the centraliser of the group \( \langle \lambda, \rho \rangle \) in the full symmetric group \( \text{Sym}(D) \) of all permutations of \( D \).
Algebraic approach to oriented maps

Pictures are disappearing ...

... and "abstract nonsense" is taking over!

Advantage: Good formalism for proofs, which we illustrate by proving Theorem 1. Recall:

**Theorem 1.**

A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup $G$ with free cyclic vertex stabilisers.

1. Fix a vertex $v$ of $\Gamma$ and define a cyclic permutation $\rho_v$ on the darts $D(v)$ at $v$ in such a way that $G \mid D(v) < \langle \rho_v \rangle$.

2. For any $g \in G$ define $\rho_g(v) = g \rho_v g^{-1}$.

3. Show that $g(v) = h(v) \Rightarrow \rho_g(v) = \rho_h(v)$.

4. Set $\rho = \prod_{g \in G^*} \rho_g(v)$ where $G^* \subset G$ is such that for each vertex $w \in \Gamma$ there is exactly one $g \in G^*$ with $g(v) = w$.

5. By no. 2, passing to $\rho$ the subscripts "disappear" and hence $\rho_g = g \rho$ for each $g \in G$.

Note that $\lambda g = g \lambda$ is automatic.

6. So, $G < \text{Aut}(M(\lambda, \rho))$, and the map $M(\lambda, \rho)$ is vertex-transitive.
Algebraic approach to oriented maps

Pictures are disappearing ...

... and “abstract nonsense” is taking over!

Theorem 1.

A connected graph \( \Gamma \) has an oriented vertex-transitive embedding if and only if \( \text{Aut}(\Gamma) \) contains a vertex-transitive subgroup \( G \) with free cyclic vertex stabilisers.

1. Fix a vertex \( v \) of \( \Gamma \) and define a cyclic permutation \( \rho_v \) on the darts \( D(v) \) at \( v \) in such a way that \( G \lhd D(v) \lhd \langle \rho_v \rangle \).

2. For any \( g \in G \) define \( \rho_g(v) = g \rho_v g^{-1} \).

3. Show that \( g(v) = h(v) \Rightarrow \rho_g(v) = \rho_h(v) \).

4. Set \( \rho = \prod_{g \in G^*} \rho_g(v) \) where \( G^* \subset G \) is such that for each vertex \( w \in \Gamma \) there is exactly one \( g \in G^* \) with \( g(v) = w \).

5. By no. 2, passing to \( \rho \) the subscripts “disappear” and hence \( \rho_g = g \rho \) for each \( g \in G \).

Note that \( \lambda g = g \lambda \) is automatic.

6. So, \( G \lhd \text{Aut}(M(\lambda, \rho)) \), and the map \( M(\lambda, \rho) \) is vertex-transitive.
... and “abstract nonsense” is taking over! Advantage: Good formalism for proofs, which we illustrate by proving Theorem 1. Recall:

**Theorem 1.**

A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup $G$ with free cyclic vertex stabilisers.

1. Fix a vertex $v$ of $\Gamma$ and define a cyclic permutation $\rho_v$ on the darts $D(v)$ at $v$ in such a way that $G | D(v) < \langle \rho_v \rangle$.

2. For any $g \in G$ define $\rho_g(v) = g \rho_v g^{-1}$.

3. Show that $g(v) = h(v) \Rightarrow \rho_g(v) = \rho_h(v)$.

4. Set $\rho = \prod_{g \in G^*} \rho_g(v)$ where $G^* \subset G$ is such that for each vertex $w \in \Gamma$ there is exactly one $g \in G^*$ with $g(v) = w$.

5. By no. 2, passing to $\rho$ the subscripts “disappear” and hence $\rho_g = g \rho$ for each $g \in G$.

Note that $\lambda g = g \lambda$ is automatic.

6. So, $G < \text{Aut}(M(\lambda, \rho))$, and the map $M(\lambda, \rho)$ is vertex-transitive.
Pictures are disappearing ...

... and “abstract nonsense” is taking over! Advantage: Good formalism for proofs, which we illustrate by proving Theorem 1. Recall:

**Theorem 1.** A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup $G$ with free cyclic vertex stabilisers.
Pictures are disappearing …

… and “abstract nonsense” is taking over! Advantage: Good formalism for proofs, which we illustrate by proving Theorem 1. Recall:

**Theorem 1.** A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup $G$ with free cyclic vertex stabilisers.

1. Fix a vertex $v$ of $\Gamma$ and define a cyclic permutation $\rho_v$ on the darts $D(v)$ at $v$ in such a way that $G_{|D(v)} < \langle \rho_v \rangle$. 

2. For any $g \in G$ define $\rho_g(v) = g \rho_v g^{-1}$.

3. Show that $g(v) = h(v) \Rightarrow \rho_g(v) = \rho_h(v)$.

4. Set $\rho = \prod_{g \in G^*} \rho_g(v)$ where $G^* \subset G$ is such that for each vertex $w \in \Gamma$ there is exactly one $g \in G^*$ with $g(v) = w$.

5. By no. 2, passing to $\rho$ the subscripts “disappear” and hence $\rho_g = g \rho$ for each $g \in G$.

6. Note that $\lambda g = g \lambda$ is automatic.

So, $G < \text{Aut}(\mathcal{M}(\lambda, \rho))$, and the map $\mathcal{M}(\lambda, \rho)$ is vertex-transitive.
Pictures are disappearing ...

... and “abstract nonsense” is taking over! Advantage: Good formalism for proofs, which we illustrate by proving Theorem 1. Recall:

**Theorem 1.** A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup $G$ with free cyclic vertex stabilisers.

1. Fix a vertex $v$ of $\Gamma$ and define a cyclic permutation $\rho_v$ on the darts $D(v)$ at $v$ in such a way that $G|_{D(v)} < \langle \rho_v \rangle$.

2. For any $g \in G$ define $\rho_g(v) = g \rho_v g^{-1}$. 
Algebraic approach to oriented maps

Pictures are disappearing ...  

... and “abstract nonsense” is taking over! Advantage: Good formalism for proofs, which we illustrate by proving Theorem 1. Recall:

Theorem 1. A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup $G$ with free cyclic vertex stabilisers.

1. Fix a vertex $v$ of $\Gamma$ and define a cyclic permutation $\rho_v$ on the darts $D(v)$ at $v$ in such a way that $G|_{D(v)} < \langle \rho_v \rangle$.

2. For any $g \in G$ define $\rho_g(v) = g\rho_v g^{-1}$.

3. Show that $g(v) = h(v) \Rightarrow \rho_g(v) = \rho_h(v)$.
Pictures are disappearing ...

... and “abstract nonsense” is taking over! Advantage: Good formalism for proofs, which we illustrate by proving Theorem 1. Recall:

**Theorem 1.** A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup $G$ with free cyclic vertex stabilisers.

1. Fix a vertex $v$ of $\Gamma$ and define a cyclic permutation $\rho_v$ on the darts $D(v)$ at $v$ in such a way that $G|_{D(v)} < \langle \rho_v \rangle$.
2. For any $g \in G$ define $\rho_g(v) = g\rho_v g^{-1}$.
3. Show that $g(v) = h(v) \Rightarrow \rho_g(v) = \rho_h(v)$.
4. Set $\rho = \prod_{g \in G^*} \rho_g(v)$ where $G^* \subset G$ is such that for each vertex $w \in \Gamma$ there is exactly one $g \in G^*$ with $g(v) = w$. 

$\rho$ is the permutation defining the embedding $\varphi$. 

By no. 2, passing to $\rho$ the subscripts “disappear” and hence $\rho_g = g\rho$ for each $g \in G$. Note that $\rho_g = g\rho$ is automatic.

So, $G < \text{Aut}(M(\lambda, \rho))$, and the map $M(\lambda, \rho)$ is vertex-transitive.
Pictures are disappearing ...

... and “abstract nonsense” is taking over! Advantage: Good formalism for proofs, which we illustrate by proving Theorem 1. Recall:

**Theorem 1.** A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup $G$ with free cyclic vertex stabilisers.

1. Fix a vertex $v$ of $\Gamma$ and define a cyclic permutation $\rho_v$ on the darts $D(v)$ at $v$ in such a way that $G_{|D(v)} < \langle \rho_v \rangle$.
2. For any $g \in G$ define $\rho_g(v) = g\rho_v g^{-1}$.
3. Show that $g(v) = h(v) \Rightarrow \rho_g(v) = \rho_h(v)$.
4. Set $\rho = \prod_{g \in G^*} \rho_g(v)$ where $G^* \subset G$ is such that for each vertex $w \in \Gamma$ there is exactly one $g \in G^*$ with $g(v) = w$.
5. By no. 2, passing to $\rho$ the subscripts “disappear” and hence $\rho g = g \rho$ for each $g \in G$. 

1. J. Širáň Open Univ. and Slovak Tech. U. Graph embeddings and symmetries 17 / 20
... and “abstract nonsense” is taking over! Advantage: Good formalism for proofs, which we illustrate by proving Theorem 1. Recall:

**Theorem 1.** A connected graph $\Gamma$ has an oriented vertex-transitive embedding if and only if $\text{Aut}(\Gamma)$ contains a vertex-transitive subgroup $G$ with free cyclic vertex stabilisers.

1. Fix a vertex $v$ of $\Gamma$ and define a cyclic permutation $\rho_v$ on the darts $D(v)$ at $v$ in such a way that $G|_{D(v)} < \langle \rho_v \rangle$.
2. For any $g \in G$ define $\rho_{g(v)} = g\rho_vg^{-1}$.
3. Show that $g(v) = h(v) \Rightarrow \rho_{g(v)} = \rho_{h(v)}$.
4. Set $\rho = \prod_{g \in G^*} \rho_{g(v)}$ where $G^* \subset G$ is such that for each vertex $w \in \Gamma$ there is exactly one $g \in G^*$ with $g(v) = w$.
5. By no. 2, passing to $\rho$ the subscripts “disappear” and hence $\rho g = g \rho$ for each $g \in G$. Note that $\lambda g = g \lambda$ is automatic.
... and “abstract nonsense” is taking over! Advantage: Good formalism for proofs, which we illustrate by proving Theorem 1. Recall:

**Theorem 1.** A connected graph \( \Gamma \) has an oriented vertex-transitive embedding if and only if \( \text{Aut}(\Gamma) \) contains a vertex-transitive subgroup \( G \) with free cyclic vertex stabilisers.

1. Fix a vertex \( v \) of \( \Gamma \) and define a cyclic permutation \( \rho_v \) on the darts \( D(v) \) at \( v \) in such a way that \( G|_{D(v)} < \langle \rho_v \rangle \).
2. For any \( g \in G \) define \( \rho_g(v) = g \rho_v g^{-1} \).
3. Show that \( g(v) = h(v) \Rightarrow \rho_g(v) = \rho_h(v) \).
4. Set \( \rho = \prod_{g \in G^*} \rho_g(v) \) where \( G^* \subset G \) is such that for each vertex \( w \in \Gamma \) there is exactly one \( g \in G^* \) with \( g(v) = w \).
5. By no. 2, passing to \( \rho \) the subscripts “disappear” and hence \( \rho g = g \rho \) for each \( g \in G \). Note that \( \lambda g = g \lambda \) is automatic.
6. So, \( G < \text{Aut}(M(\lambda, \rho)) \), and the map \( M(\lambda, \rho) \) is vertex-transitive. \( \square \)
Cayley maps

A “cheating” way of constructing vertex-transitive graphs:

Given a group $G$ and a generating set $X$ of $G$ such that $X^{-1} = X$, the Cayley graph $\Gamma = \text{Cay}(G, X)$ has vertex set $G$ and dart set $D = \{(g, x); g \in G, x \in X\}$. A dart $(g, x)$ emanates from $g$ and terminates at $gx$. Note that $(gx, x^{-1})$ is the reverse dart to $(g, x)$; this pair forms an undirected edge of $\Gamma$. Therefore, $\lambda(g, x) = (gx, x^{-1})$.

Same type of “cheating” applies to constructing vertex-transitive maps: Let $\pi$ be any cyclic permutation of $X$. Define $\rho$ on $D$ by $\rho(g, x) = (g, \pi(x))$. The map $M = M(\lambda, \rho)$ is called a Cayley map. A Cayley map can be loosely described as an oriented embedding of a Cayley graph in which the cyclic order of darts (in terms of generators) at any vertex is “the same”.
Cayley maps

A “cheating” way of constructing vertex-transitive graphs:

Given a group $G$ and a generating set $X$ of $G$ such that $X^{-1} = X$,
A “cheating” way of constructing vertex-transitive graphs:

Given a group $G$ and a generating set $X$ of $G$ such that $X^{-1} = X$, the Cayley graph $\Gamma = \text{Cay}(G, X)$ has vertex set $G$ and dart set $D = \{(g, x); \ g \in G, \ x \in X\}$. 
Cayley maps

A “cheating” way of constructing vertex-transitive graphs:

Given a group $G$ and a generating set $X$ of $G$ such that $X^{-1} = X$, the Cayley graph $\Gamma = \text{Cay}(G, X)$ has vertex set $G$ and dart set $D = \{(g, x); \ g \in G, \ x \in X\}$. A dart $(g, x)$ emanates from $g$ and terminates at $gx$. 

Note that $(gx, x^{-1})$ is the reverse dart to $(g, x)$; this pair forms an undirected edge of $\Gamma$. Therefore, $\lambda(g, x) = (gx, x^{-1})$.

Same type of “cheating” applies to constructing vertex-transitive maps: Let $\pi$ be any cyclic permutation of $X$. Define $\rho$ on $D$ by $\rho(g, x) = (g, \pi(x))$. The map $M = M(\lambda, \rho)$ is called a Cayley map. A Cayley map can be loosely described as an oriented embedding of a Cayley graph in which the cyclic order of darts (in terms of generators) at any vertex is “the same”.

Cayley maps

A “cheating” way of constructing vertex-transitive graphs:

Given a group $G$ and a generating set $X$ of $G$ such that $X^{-1} = X$, the Cayley graph $\Gamma = \text{Cay}(G, X)$ has vertex set $G$ and dart set $D = \{(g, x) ; g \in G, x \in X\}$. A dart $(g, x)$ emanates from $g$ and terminates at $gx$. Note that $(gx, x^{-1})$ is the reverse dart to $(g, x)$; this pair forms an undirected edge of $\Gamma$. 
A “cheating” way of constructing vertex-transitive graphs:

Given a group $G$ and a generating set $X$ of $G$ such that $X^{-1} = X$, the Cayley graph $\Gamma = \text{Cay}(G, X)$ has vertex set $G$ and dart set $D = \{(g, x); \ g \in G, \ x \in X\}$. A dart $(g, x)$ emanates from $g$ and terminates at $gx$. Note that $(gx, x^{-1})$ is the reverse dart to $(g, x)$; this pair forms an undirected edge of $\Gamma$. Therefore, $\lambda(g, x) = (gx, x^{-1})$. 

Same type of “cheating” applies to constructing vertex-transitive maps:

Let $\pi$ be any cyclic permutation of $X$. Define $\rho$ on $D$ by $\rho(g, x) = (g, \pi(x))$. The map $M = M(\lambda, \rho)$ is called a Cayley map. A Cayley map can be loosely described as an oriented embedding of a Cayley graph in which the cyclic order of darts (in terms of generators) at any vertex is “the same.”
A “cheating” way of constructing vertex-transitive graphs:

Given a group $G$ and a generating set $X$ of $G$ such that $X^{-1} = X$, the Cayley graph $\Gamma = \text{Cay}(G, X)$ has vertex set $G$ and dart set $D = \{(g, x); \ g \in G, \ x \in X\}$. A dart $(g, x)$ emanates from $g$ and terminates at $gx$. Note that $(gx, x^{-1})$ is the reverse dart to $(g, x)$; this pair forms an undirected edge of $\Gamma$. Therefore, $\lambda(g, x) = (gx, x^{-1})$.

Same type of “cheating” applies to constructing vertex-transitive maps:
A “cheating” way of constructing vertex-transitive graphs:

Given a group $G$ and a generating set $X$ of $G$ such that $X^{-1} = X$, the Cayley graph $\Gamma = \text{Cay}(G, X)$ has vertex set $G$ and dart set $D = \{(g, x); \; g \in G, \; x \in X\}$. A dart $(g, x)$ emanates from $g$ and terminates at $gx$. Note that $(gx, x^{-1})$ is the reverse dart to $(g, x)$; this pair forms an undirected edge of $\Gamma$. Therefore, $\lambda(g, x) = (gx, x^{-1})$.

Same type of “cheating” applies to constructing vertex-transitive maps:

Let $\pi$ be any cyclic permutation of $X$. Define $\rho$ on $D$ by $\rho(g, x) = (g, \pi(x))$. The map $M = M(\lambda, \rho)$ is called a Cayley map.
Cayley maps

A “cheating” way of constructing vertex-transitive graphs:

Given a group $G$ and a generating set $X$ of $G$ such that $X^{-1} = X$, the Cayley graph $Γ = \text{Cay}(G, X)$ has vertex set $G$ and dart set $D = \{(g, x); \ g \in G, \ x \in X\}$. A dart $(g, x)$ emanates from $g$ and terminates at $gx$. Note that $(gx, x^{-1})$ is the reverse dart to $(g, x)$; this pair forms an undirected edge of $Γ$. Therefore, $λ(g, x) = (gx, x^{-1})$.

Same type of “cheating” applies to constructing vertex-transitive maps:

Let $\pi$ be any cyclic permutation of $X$. Define $ρ$ on $D$ by $ρ(g, x) = (g, \pi(x))$. The map $M = M(λ, ρ)$ is called a Cayley map.

A Cayley map can be loosely described as an oriented embedding of a Cayley graph in which the cyclic order of darts (in terms of generators) at any vertex is “the same”.
Example. $K_5$ on a torus as a Cayley map for the group $\mathbb{Z}_5$, generating set $X = \{1, 2, 3, 4\}$, with $\pi = (1, 3, 4, 2)$:
Example. $K_5$ on a torus as a Cayley map for the group $\mathbb{Z}_5$, generating set $X = \{1, 2, 3, 4\}$, with $\pi = (1, 3, 4, 2)$:
Example. \( K_5 \) on a torus as a Cayley map for the group \( \mathbb{Z}_5 \), generating set \( X = \{1, 2, 3, 4\} \), with \( \pi = (1, 3, 4, 2) \):

Observe that Cayley maps are automatically vertex-transitive.
Example. $K_5$ on a torus as a Cayley map for the group $\mathbb{Z}_5$, generating set $X = \{1, 2, 3, 4\}$, with $\pi = (1, 3, 4, 2)$:

Observe that Cayley maps are automatically vertex-transitive. Indeed, it can be checked that for any $h \in G$ the mapping $A_h$ defined by $A_h(g, x) = (hg, x)$ is in $\text{Aut}(M)$. 
Example. $K_5$ on a torus as a Cayley map for the group $\mathbb{Z}_5$, generating set $X = \{1, 2, 3, 4\}$, with $\pi = (1, 3, 4, 2)$:

Observe that Cayley maps are automatically vertex-transitive. Indeed, it can be checked that for any $h \in G$ the mapping $A_h$ defined by $A_h(g, x) = (hg, x)$ is in $\text{Aut}(M)$.

The group $\{A_h; h \in G\} \cong G$ is regular (i.e., transitive and free) on vertices of the Cayley map.
Example. $K_5$ on a torus as a Cayley map for the group $\mathbb{Z}_5$, generating set $X = \{1, 2, 3, 4\}$, with $\pi = (1, 3, 4, 2)$:

Observe that Cayley maps are automatically vertex-transitive. Indeed, it can be checked that for any $h \in G$ the mapping $A_h$ defined by $A_h(g, x) = (hg, x)$ is in $\text{Aut}(M)$. The group $\{A_h; h \in G\} \cong G$ is regular (i.e., transitive and free) on vertices of the Cayley map. So, $G$ is “just big enough” to make the Cayley map vertex-transitive.
Example. $K_5$ on a torus as a Cayley map for the group $\mathbb{Z}_5$, generating set $X = \{1, 2, 3, 4\}$, with $\pi = (1, 3, 4, 2)$:

Observe that Cayley maps are automatically vertex-transitive. Indeed, it can be checked that for any $h \in G$ the mapping $A_h$ defined by $A_h(g, x) = (hg, x)$ is in $\text{Aut}(M)$.

The group $\{A_h; h \in G\} \cong G$ is regular (i.e., transitive and free) on vertices of the Cayley map. So, $G$ is “just big enough” to make the Cayley map vertex-transitive.

If there are no other automorphisms, such Cayley maps can be viewed as vertex-transitive maps with the lowest “level of symmetry”.

[Diagram of a Cayley map on a torus]
Conclusion and research problems

We have encountered vertex-transitive embeddings, among which the Cayley maps had the “lowest level of symmetry”. Similar theory has been developed for edge-transitive maps. In the second lecture we will continue with considering the “highest level of symmetry” in oriented maps.

PROBLEMS

Warm-up: Find an infinite class of graphs $\Gamma$ such that $\text{Aut}(\Gamma)$ has a vertex-transitive subgroup with cyclic vertex stabilisers which are not free in their action on darts at a vertex.

Potential paper: Investigate spectra of genera of surfaces admitting oriented vertex-transitive embeddings of graphs. For which “interesting” graphs is such a spectrum a non-trivial interval?

Potential thesis: Develop a theory for (oriented as well as unoriented) face-transitive embeddings of graphs.
Conclusion and research problems

We have encountered vertex-transitive embeddings, among which the Cayley maps had the “lowest level of symmetry”.

Similar theory has been developed for edge-transitive maps. In the second lecture we will continue with considering the “highest level of symmetry” in oriented maps.

PROBLEMS

Warm-up: Find an infinite class of graphs $\Gamma$ such that $\text{Aut}(\Gamma)$ has a vertex-transitive subgroup with cyclic vertex stabilisers which are not free in their action on darts at a vertex.

Potential paper: Investigate spectra of genera of surfaces admitting oriented vertex-transitive embeddings of graphs. For which “interesting” graphs is such a spectrum a non-trivial interval?

Potential thesis: Develop a theory for (oriented as well as unoriented) face-transitive embeddings of graphs.
Conclusion and research problems

We have encountered vertex-transitive embeddings, among which the Cayley maps had the “lowest level of symmetry”. Similar theory has been developed for edge-transitive maps.
Conclusion and research problems

We have encountered vertex-transitive embeddings, among which the Cayley maps had the “lowest level of symmetry”. Similar theory has been developed for edge-transitive maps. In the second lecture we will continue with considering the “highest level of symmetry” in oriented maps.
Conclusion and research problems

We have encountered vertex-transitive embeddings, among which the Cayley maps had the “lowest level of symmetry”. Similar theory has been developed for edge-transitive maps. In the second lecture we will continue with considering the “highest level of symmetry” in oriented maps.

PROBLEMS

Warm-up: Find an infinite class of graphs $\Gamma$ such that $\text{Aut}(\Gamma)$ has a vertex-transitive subgroup with cyclic vertex stabilisers which are not free in their action on darts at a vertex.

Potential paper: Investigate spectra of genera of surfaces admitting oriented vertex-transitive embeddings of graphs. For which “interesting” graphs is such a spectrum a non-trivial interval?

Potential thesis: Develop a theory for (oriented as well as unoriented) face-transitive embeddings of graphs.
Conclusion and research problems

We have encountered vertex-transitive embeddings, among which the Cayley maps had the “lowest level of symmetry”. Similar theory has been developed for edge-transitive maps. In the second lecture we will continue with considering the “highest level of symmetry” in oriented maps.

PROBLEMS

Warm-up: Find an infinite class of graphs $\Gamma$ such that $\text{Aut}(\Gamma)$ has a vertex-transitive subgroup with cyclic vertex stabilisers which are not free in their action on darts at a vertex.

Potential paper: Investigate spectra of genera of surfaces admitting oriented vertex-transitive embeddings of graphs. For which “interesting” graphs is such a spectrum a non-trivial interval?

Potential thesis: Develop a theory for (oriented as well as unoriented) face-transitive embeddings of graphs.
Conclusion and research problems

We have encountered vertex-transitive embeddings, among which the Cayley maps had the “lowest level of symmetry”. Similar theory has been developed for edge-transitive maps. In the second lecture we will continue with considering the “highest level of symmetry” in oriented maps.

PROBLEMS

Warm-up: Find an infinite class of graphs $\Gamma$ such that $\text{Aut}(\Gamma)$ has a vertex-transitive subgroup with cyclic vertex stabilisers which are not free in their action on darts at a vertex.

Potential paper: Investigate spectra of genera of surfaces admitting oriented vertex-transitive embeddings of graphs. For which “interesting” graphs is such a spectrum a non-trivial interval?
Conclusion and research problems

We have encountered vertex-transitive embeddings, among which the Cayley maps had the “lowest level of symmetry”. Similar theory has been developed for edge-transitive maps. In the second lecture we will continue with considering the “highest level of symmetry” in oriented maps.

PROBLEMS

Warm-up: Find an infinite class of graphs $\Gamma$ such that $\text{Aut}(\Gamma)$ has a vertex-transitive subgroup with cyclic vertex stabilisers which are not free in their action on darts at a vertex.

Potential paper: Investigate spectra of genera of surfaces admitting oriented vertex-transitive embeddings of graphs. For which “interesting” graphs is such a spectrum a non-trivial interval?

Potential thesis: Develop a theory for (oriented as well as unoriented) face-transitive embeddings of graphs.