18.03 Practice Hour Exam III, April, 2006

Properties of the Laplace transform

0. Definition: \[ \mathcal{L}[f(t)] = F(s) = \int_{0}^{\infty} f(t) e^{-st} dt \] for Re \( s \gg 0 \).

1. Linearity: \[ \mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s) \]

2. Inverse transform: \( F(s) \) essentially determines \( f(t) \).

3. \( s \)-shift rule: \[ \mathcal{L}[e^{at}f(t)] = F(s-a) \]

4. \( t \)-shift rule: \[ \mathcal{L}[f(a)(t)] = e^{-as}F(s), \quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases} \]

5. \( s \)-derivative rule: \[ \mathcal{L}[tf(t)] = -F'(s) \]

6. \( t \)-derivative rule: \[ \mathcal{L}[f'(t)] = sF(s) - f(0+) \]
\[ \mathcal{L}[f''(t)] = s^2F(s) - sf(0+) - f'(0+) \]
where we ignore singularities in derivatives at \( t = 0 \).

7. Convolution rule: \[ \mathcal{L}[f(t) * g(t)] = F(s)G(s), \quad f(t) * g(t) = \int_{0}^{t} f(\tau)g(t-\tau)d\tau \]

8. Weight function: \[ \mathcal{L}[w(t)] = W(s) = 1/p(s), \quad w(t) \text{ the unit impulse response.} \]

Formulas for the Laplace transform

\[ \mathcal{L}[1] = 1/s, \quad \mathcal{L}[e^{at}] = 1/(s-a) \]
\[ \mathcal{L}[\cos(\omega t)] = s/(s^2 + \omega^2), \quad \mathcal{L}[\sin(\omega t)] = \omega/(s^2 + \omega^2) \]
\[ \mathcal{L}[u_a(t)] = e^{-as}/s, \quad \mathcal{L}[\delta_a(t)] = e^{-as} \]
\[ \mathcal{L}[t^n] = n!/s^{n+1} \]

Fourier coefficients

\[ f(t) = a_0/2 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + \cdots \]
\[ a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(mt) \, dt, \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(mt) \, dt \]

If \( \text{sq}(t) \) is the odd function of period \( 2\pi \) which has value 1 between 0 and \( \pi \), then
\[ \text{sq}(t) = \frac{4}{\pi} \left( \sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots \right) . \]
First Practice Exam

1. (a) $p(D)$ is an LTI operator, and what we know about it is its weight function (or unit impulse response) $w(t)$. Express the solution to $p(D)x = \sin t$ with rest initial conditions as an integral involving $w(t)$.

(b) and (c) involve a certain LTI differential operator $p(D)$ has weight function (or unit impulse response) given by $w(t) = u(t)e^{-t}\sin(t)$.

(b) What is the corresponding transfer function $W(s)$?

(c) What is the exponential solution of the equation $p(D)x = e^{-2t}$?

2. In this problem, $X(s) = \frac{4}{s(s^2 + 2s + 2)}$.

(a) Find a function $x(t)$ having Laplace transform $X(s)$.

(b) Sketch the pole diagram of $X(s)$. Shade the region in which the integral definition of the Laplace transform of $x(t)$ converges.

3. What is the Laplace transform of the solution $x(t)$ to $2\ddot{x} + 3x = \sin t + \delta(t - \pi)$ with initial condition $x(0) = 1$? (You are not asked to solve the differential equation.)

4. (a) Find the Fourier series for the function $f(t)$ which is periodic of period 4 and such that $f(t) = \begin{cases} 1 & \text{for } -1 < t < 1, \\ 0 & \text{for } 1 < t < 3. \end{cases}$

5. In this problem $f(t) = \sum_{k=1}^{\infty} \frac{\sin(2kt)}{2^k} = \frac{\sin(2t)}{2} + \frac{\sin(4t)}{4} + \frac{\sin(6t)}{8} + \cdots$.

(a) Find the Fourier series expression for a periodic solution $x_p$ to $\ddot{x} + \omega_n^2 x = f(t)$.

(b) For what values of $\omega_n$ (if any) do there fail to be periodic solutions?

(c) Write down a solution to $\ddot{x} + 4x = f(t)$.

Second Practice Exam

1. Let $p(D)$ be the LTI differential operator with transfer function $W(s) = \frac{1}{s^2 + 4s + 8}$.

(a) What is the characteristic polynomial $p(s)$?

(b) What is the weight function (or unit impulse response) $w(t)$ of this operator $p(D)$?

2. $p(D)$ will continue to be the differential operator with transfer function $W(s) = \frac{1}{s^2 + 4s + 8}$.

(a) What is the Laplace transform $X(s)$ of the solution to $p(D)x = e^t$ with initial conditions $x(0) = 1$, $\dot{x}(0) = 3$? (You are not asked to find the solution itself!)

(b) Express the solution to $p(D)x = \delta(t - 1)$ with rest initial conditions in terms of the weight function $w$.

3. This problem deals with $X(s) = \frac{1}{s(s^2 + 4s + 8)}$. 
(a) What function \( x(t) \) has Laplace transform \( X(s) \)?

(b) Write down an initial value problem whose solution is this function \( x(t) \) for \( t > 0 \). (Don’t neglect the initial condition!)

4. (a) Still with regard to the function \( X(s) = \frac{1}{s(s^2 + 4s + 8)} \):
   Sketch its pole diagram, and shade in the region where the integral expressing it as the Laplace transform of \( x(t) \) converges.

(b) New topic: Compute the convolution product \( t \ast t \).

5. (a) What is the Fourier series of the function \( 1 + \text{sq}(2t) \), where \( \text{sq}(t) \) is the standard squarewave (described on the attached information sheet)?

(b) What is the Fourier series of the generalized function which is odd, of period \( 2\pi \), and between 0 and \( \pi \) is given by \( \delta(t - \pi/2) \)?

6. The sawtooth function of period 2, given by \( f(t) = |t| \) for \( t \) between \(-1 \) and \( +1 \), has Fourier series

\[
\sum_{k=1}^{\infty} \frac{\cos(kt)}{k^2}.
\]

(a) For a general constant \( \omega_n \geq 0 \), what is the Fourier series of a periodic solution to \( \ddot{x} + \omega_n^2 x = f(t) \)?

(b) For what values of \( \omega_n \) is the system in resonance with this signal?

**Solutions to First Practice Exam**

1. (a) \( x(t) = \sin(t) \ast w(t) = \int_0^t \sin(u) w(t-u) \, du \).

(b) \( \mathcal{L}[w(t)] = \frac{1}{(s + 1)^2 + 1} = \frac{1}{s^2 + 2s + 2} \).

(c) The transfer function evaluated at \(-2\) gives the multiple, by the Exponential Response formula: \( W(-2) = 1/2 \), so \( x_p = (1/2)e^{-2t} \).

2. (a) \( \frac{4}{s(s^2 + 2s + 2)} = \frac{a}{s} + \frac{b(s + 1) + c}{(s + 1)^2 + 1} \). Cover up the \( s \) and set \( s = 0 \) to see \( 4/2 = a \).
   Cover up the \( (s + 1)^2 + 1 \) and set \( s = -1 + i \) (i.e. \( s + 1 = i \)) to see \( 4/(-1 + i) = bi + c \), i.e. \( 4(-1 - i)/2 = bi + c \) or \( b = -2, c = -2 \). So \( X(s) = \frac{2}{s} - 2 \frac{(s + 1) + 1}{(s + 1)^2 + 1} \), which is the Laplace transform of \( 2 - 2e^{-t}(\cos(t) + \sin(t)) \).

(b) The poles occur at \( s = 0 \) and at \( s = -1 \pm i \). The region of convergence is the right half plane.

3. (a) \( 2(X(s) - 1) + 3X(s) = \frac{1}{s^2 + 1} + e^{-\pi s} \), so \( X = \frac{2 + 1/(s^2 + 1) + e^{-\pi s}}{2s + 3} \).

4. \( f(t) = \frac{1}{2} + \frac{1}{2} \text{sq} \left( \frac{\pi t}{2} + \frac{\pi}{2} \right) = \frac{1}{2} + \frac{4}{\pi} \left( \sin((\pi t/2) + (\pi/2)) + \frac{\sin((3\pi t/2) + (3\pi/2))}{3} + \cdots \right) \)
   \[ = \frac{1}{2} + \frac{2}{\pi} \left( \cos(\pi t/2) - \frac{\cos(3\pi t/2)}{3} + \frac{\cos(5\pi t/2)}{5} - \cdots \right) \]
5. (a) \[ x_p = \sum_{k=1}^{\infty} \frac{\sin(2kt)}{2^k(\omega_n^2 - 4k^2)}. \]

(b) \( \omega_n = 2, 4, 6, \ldots. \)

(c) The equation \( \ddot{z} + 4z = e^{2it} \) exhibits resonance. The characteristic polynomial \( p(s) = s^2 + 4 \) has derivative \( p'(s) = 2s \), and \( p'(2i) = 4i \), so \( z_p = \frac{te^{2it}}{4t} \) and \( \ddot{x} + 4x = \sin(2t) \) has solution \( \text{Im} (z_p) = -\frac{t}{4} \cos(2t) \). Thus \( x_p = -\frac{t}{8} \cos(2t) + \sum_{k=2}^{\infty} \frac{\sin(2kt)}{2^k(4 - 4k^2)}. \)

Solutions to Second Practice Exam

1. (a) \( p(s) = s^2 + 4s + 8. \)

(b) \( W(s) = 1/(s^2 + 4s + 8) = (1/2)(2/((s + 2)^2 + 4) \) is the Laplace transform of \( w(t) = (1/2)e^{-2t}\sin(2t) \) (for \( t > 0 \); if you have to give it a value for \( t < 0 \), it is zero).

2. (a) \( (s^2X(s) - s - 3) + 4(sX(s) - 1) + 8X(s) = 1/(s-1), \) so \( X(s) = \frac{(s+7) + 1/(s-1)}{s^2 + 4s + 8}. \)

(b) \( x(t) = w(t-1). \)

3. (a) \( X(s) = 1/(s(s^2 + 4s + 8)) = a/s + (b(s + 2) + c)/((s + 2)^2 + 4). \) Multiply through by \( s \) and set \( s = 0 \) to see \( a = 1/8. \) Multiply through by \( (s + 2)^2 + 4 \) and set \( s = -2 + 2i \) to see \( b(2i) + c = 1/(-2 + 2i) = (-1 - i)/4 \) or \( b = -1/8, c = -1/4: \) so \( X(s) = (1/8)(1/s - (1/8)(s + 2)/(s + 2)^2 + 4) - (1/2)/(s + 2)^2 + 4, \) which is the Laplace transform of \( (1/8)(1 - e^{-2t}\cos(2t) + \sin(2t)). \)

(b) This problem has many answers. A really cheap one is \( x(t) = (1/8)(1 - e^{-2t}\cos(2t) + \sin(2t)): \) this is a “zeroth order” ODE. Slightly trickier would be to write \( \dot{x} \) the derivative of this, with initial condition \( x(0) = 0. \) These are correct and so acceptable answers, but more expected answers are: \( \ddot{x} + 4\dot{x} + 8x = 1 \) with rest initial conditions (since the standard method of solving \( p(D)x = 1 \) with rest initial conditions leads to \( X(s) = 1/(s p(s)), \) or (using 1(b)) \( \dot{x} = (1/2)e^{-2t}\sin(2t) \) with rest initial conditions.

4. (a) There are poles at \( s = 0 \) and at \( s = -2 \pm 2i. \) The integral converges for \( \text{Re} (s) > 0, \) to the right of the vertical line through 0.

(b) \( t * t = \int_{0}^{t} u(t-u) \, du = [tu^2/2 - u^3/3]_{0}^{t} = t^3/6. \) Alternate solution: \( L[t] = 1/s^2 \) so \( L[t * t] = 1/s^4, \) which is the Laplace transform of \( t^3/6. \)

5. (a) \( 1 + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(2nt)}{n}. \)

(b) The function is odd so \( a_n = 0 \) for all \( n. \) \( b_n = (2/\pi) \int_{0}^{\pi} \delta(t - \pi/2) \sin(nt) \, dt = (2/\pi) \sin(\pi n/2). \) \( \sin(\pi n/2) = 0 \) for \( n \) even, and for odd \( n \) alternates between the values 1 and -1. Thus the series is \( (2/\pi)(\sin(t) - \sin(3t) + \sin(5t) - \sin(7t) + \cdots). \)

6. (a) \( \frac{1}{2\omega_n^2} - \frac{4}{\pi^2} \sum_{k \text{ odd}} \frac{\cos(knt)}{k^2(\omega_n^2 - (k\pi)^2)}. \)

(b) Resonance occurs for \( \omega_n = n\pi, \) where \( n \) is a positive odd integer.