1RSB in the ‘small-world’ spin glass

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Introduction and motivation

*In what sense small world?*
Ising spins interacting through

- Ferromagnetic short range interactions $J_0$ on a ring
- Frustrated long range interactions $J_1$ as "shortcuts".
Why bother?

1 Spin glass study with finite connectivity some kind of geometry ("maximal tractable model")
2 RKKY interactions (short range ferro, long range oscillatory)
3 Technical motivations:
   – Extending previous RS work which used replicated transfer matrices
   – The cavity method gives a more intuitive interpretation of replica results
   – RSB exists at least in special cases of the model
   – Conjecture in RS paper on non-reentrance in the phase diagram
   – Possible consequences for message passing algorithms on small world type networks?
Definitions

- Hamiltonian:
  
  \[ H = -J_0 \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - \frac{1}{\langle k \rangle} \sum_{i,j=0}^{N} J_{ij} c_{ij} \sigma_i \sigma_j \]

  where \( \sigma_i \in \{-1, 1\} \) and \( \sigma_{N+1} \equiv \sigma_1 \).

- Random variables:

  \[ p(J_{ij}) \rightarrow J_{ij} \]

  \[ p(k_i) \rightarrow c_{ij} \in \{0, 1\} \]

  through the constraint \( \sum_{j=1}^{N} c_{ij} = k_i \).

  Furthermore, \( c_{ii} = 0 \), \( \langle k \rangle = \sum_k p(k) k \)

- Goal: obtaining phase diagrams through calculation of observables

  \( m = \frac{1}{N} \sum_i \langle \sigma_i \rangle \) and \( q = \frac{1}{N} \sum_i \langle \sigma_i \rangle \langle \sigma_i \rangle \)
Dealing with the randomness

Objective:

\[ f = \lim_{N \to \infty} \frac{1}{\beta N} \left\langle \log Z \right\rangle_{J,k} \]

where

\[ Z = \sum_{\sigma} e^{-\beta H(\sigma, \{J_{ij}, \{k_i\})} \]

1 *Replica theory:*

\[ f = \lim_{N \to \infty} \lim_{n \to 0} \frac{1}{\beta N} \log \left\langle Z^n \right\rangle_{J,k} \]

Solution (RS) using replicated transfer matrices

2 *Cavity method:* Local iteration equations with random disorder.
The Cavity method on a random graph

- Basic (RS) assumption: All $k$ neighbours of spin $\sigma_0$ are only statistically dependent through spin $\sigma_0$ (no loop effects).
- In absence of spin $\sigma_0$, state of neighbouring spin $\sigma_j$ (then having $k - 1$ neighbours) is characterized by

$$p(\sigma_j) \sim e^{\beta h_j \sigma_j}$$
• Upon linking spin \( \sigma_0 \) with \( k - 1 \) neighbours, one may define

\[
Z(\sigma_0) = \sum_{\sigma_1, \ldots, \sigma_{k-1}} \exp \left\{ \beta \left[ \sigma_0 \sum_{l=1}^{k-1} J_l \sigma_l + \sum_{l=1}^{k-1} h_l \sigma_l \right] \right\}
\]

\[
= \frac{\exp \left\{ \beta \sigma_0 \sum_{l=1}^{k-1} u(J_l, h_l) \right\}}{c(\{J_l\}, \{h_l\})}
\]

with

\[
u(J_l, h_l) = \frac{1}{\beta} \tanh^{-1} [\tanh(\beta J_l) \tanh(\beta h_l)]
\]

• It follows that

\[h_0 = \sum_{l=1}^{k-1} u(J_l, h_l)\]

• The iterative equation of cavity field distributions for fixed connectivity \( k \)

\[
W(h) = \int \prod_{l=1}^{k-1} \left[ dJ_l p(J_l) dh_l W(h_l) \right] \delta \left[ h - \sum_{l=1}^{k-1} u(J_l, h_l) \right]
\]
Interpretation of $h_0$ and $u_i$: they parametrize messages $\mu_{0 \to (0j)}(\sigma_0) \sim e^{\beta h_0 \sigma_0}$, $\mu_{(i0) \to 0}(\sigma_0) \sim e^{\beta u_i \sigma_0}$. Consequently, $W(h)$ parametrizes the distribution of messages over the whole graph, i.e., describing statistics of belief propagation over random instances of graphs.

- $W(h)$ is the main order parameter of the RS theory, from which all macroscopic observables follow.
Cavity method for the ‘small-world’ lattice

- Two types of cavity spins, either missing a long-range (τ-spin), or a short-range bond (σ-spin).
- Two types of cavity fields, $h$ (coupling to τ-spin) and $x$ (coupling to σ-spin).

Two types of iterations: Adding to the graph $\sigma_0$ (left) or $\tau_0$ (right).
\[ x_0 = u(J_0, x_1) + \sum_{l=1}^{k} u(J_l, h_l) \]

\[ h_0 = u(J_0, x_1) + u(J_0, x_2) + \sum_{l=1}^{k-1} u(J_l, h_l) \]

- Coupled distribution iterations

\[ \Phi(x) = \int \, dx' \Phi(x') \prod_{l=1}^{k} [dJ_l p(J_l) dh_l W(h_l)] \]

\[ \times \delta[x - u(J_0, x') - \sum_{l=1}^{k} u(J_l, h_l)] \]

\[ W(h) = \int \, dx \, dx' \Phi(x) \Phi(x') \prod_{l=1}^{k-1} [dJ_l p(J_l) dh_l W(h_l)] \]

\[ \times \delta[h - u(J_0, x) - u(J_0, x') - \sum_{l=1}^{k-1} u(J_l, h_l)] \]
• ‘Real effective field’ distribution:

\[ R(H) = \int dx dx' \Phi(x) \Phi(x') \prod_{l=1}^{k} [dJ_l p(J_l) dh_l W(h_l)] \times \delta[H - u(J_0, x) - u(J_0, x') - \sum_{l=1}^{k} u(J_l, h_l)] \]

• Observables:

\[ m = \int dH R(H) \tanh(\beta H) \]

\[ q = \int dH R(H) \tanh^2(\beta H) \]

• Other observables are found by defining appropriate graph operations from which they are derived.

• Resulting equations equivalent to replicated transfer matrix results
Bifurcation conditions and RS phase diagrams

- Bifurcation conditions for the first moments of $W(h)$ determine second order phase transitions from paramagnetic phase:
  
  Paramagnetic ($m = 0, q = 0$) to Ferromagnetic ($m \neq 0, q \neq 0$):
  
  \[
  \Bigg[ \langle k \rangle (e^{2\beta J_0} - 1) + \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \Bigg] \langle \tanh \left( \frac{\beta J}{\langle k \rangle} \right) \rangle_J = 1
  \]

  Paramagnetic to Spin Glass ($m = 0, q \neq 0$):
  
  \[
  \Bigg[ 2\langle k \rangle \sinh^2(\beta J_0) + \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \Bigg] \langle \tanh^2 \left( \frac{\beta J}{\langle k \rangle} \right) \rangle_J = 1
  \]
One step RSB

- RS assumption: cavity fields characterize global minimum of free energy before and after graph iteration with corresponding free energy shift $\Delta F$.
- 1RSB (Mezard & Parisi) Several local minima are considered characterized by $h^\alpha$ ($\alpha$ labels pure state). Corresponding free energy shifts $\Delta F^\alpha$ reshuffle order in $F^\alpha$.
- Different pure states $\alpha \in \{1, \ldots, M\}$

$$W^\alpha = \frac{\exp(-\beta F^\alpha)}{\sum_{\gamma} \exp(-\beta F^\gamma)}$$

- Basic self-consistent ansatz

$$\rho(F) = \exp(\beta \mu (F - F^{\text{ref}}))$$

- Order parameter functions site-dependent and factorize over pure states at each
\begin{equation}
P(h) = \frac{1}{N} \sum_i \prod_{\alpha=1}^{\mathcal{M}} P_i(h^\alpha)
\end{equation}

\begin{equation}
Q(x) = \frac{1}{N} \sum_i \prod_{\alpha=1}^{\mathcal{M}} Q_i(x^\alpha)
\end{equation}

- Advanced population dynamics algorithm, involving $N$ populations of $\mathcal{M}$ fields.
- All observables are evaluated at value of $\mu$ for which

\[
\frac{\partial f}{\partial \mu} = 0
\]

Equations are recovered exactly within replica theory using a one step RSB ansatz a la Monasson.
Numerical results

\[ p(k) = \delta_{k,6}, \]

\[ p(J) = \frac{5}{8}\delta(J - 1) + \frac{3}{8}(J + 1) \]
RS:
\[ f = -0.3561 \]
\[ q = 0.5789 \]

1RSB:
\[ \mathcal{N} = 2000 \text{ and } \mathcal{M} = 1000 \]
\[ \mu = 0.32 \pm 0.01. \]
\[ f = -0.3557 \pm 0.0001 \]
\[ q_0 = 0.397 \pm 0.003 \]
\[ q_1 = 0.673 \pm 0.003 \]
\[ m = 0.2 \pm 0.05 \]
Simulations

Unfortunately: large finite size effects, long equilibration times. Bad statistics. However: nonzero magnetization.
Conclusions and outlook

- Small world type graph can be studied with the cavity method, giving the same results as the replica method.
- RSB occurs in various regions of the phase diagram, and can be detected within this framework.
- Extension to next-nearest neighbour interactions possible, though numerically expensive.