E-team on Unsupervised Segmentation
Fully Bayesian Source Separation with Application to the Cosmic Microwave Background

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The Problem — Factor Analysis

- $J$ observations of vectors $d_1, \ldots, d_J$, $d_j \in \mathbb{R}^{n_f}$;
- Assume:
  \[ d_j = A s_j + \epsilon_j, \quad s_j \in \mathbb{R}^{n_s}, \]
  where $A$ is a $n_f \times n_s$ matrix (the mixing matrix);
- Assume $\epsilon_j \sim N(0, \text{diag}(\tau_1^{-1}, \ldots, \tau_{n_f}^{-1}))$;
- Goal: observe the $d_j$ and then infer $A$ and recover ("separate") the $s_j$;
Cosmic Microwave Background (CMB)

- Discovered by accident in 1964;
- By 1970’s agreed to be an image of the first scattering of EM radiation at recombination $\approx 300,000$ years after Big Bang;
- Of great interest as an observation of the state of the early universe:
  - In particular it is remarkably uniform;
  - But accurate measurement of the small anisotropies place strong restrictions on theories of big bang, galaxy formation etc.;
- Cosmic expansion $\Rightarrow$ radiation has cooled to 2.7K (microwave);
CMB Spectrum — Black Body

Theory and observation agree
Inferring the CMB — Source Separation
Separating the Cosmic Microwave Background

- $d_j$, are observations at $J$ pixels over the sky at $n_f$ microwave frequencies $\nu_1, \ldots, \nu_{n_f}$;
  - Upcoming data (Planck satellite) will have $J \approx 10^7$, $n_f = 9$ at frequencies from 30 to 857 GHz;
- $s_j$ are the sources that make up the microwave received by the satellite:
  - One of these sources is the CMB (source 1);
  - Other important ones are synchrotron radiation and galactic dust;
  - There are others.... is $n_s$ known?
  - A lot is known from physics about the properties of these sources e.g. their spectrum, mean, variance etc;
- $A$ is not known but the physics tell us a lot about it;
- A lot of “prior” information $\Rightarrow$ a Bayesian approach looks promising.
We can put all the $d_j$ and $s_j$ into matrices:

\[
D = \{d_{ij} \mid i = 1, \ldots, n_f, j = 1, \ldots, J\};
\]
\[
S = \{s_{kj} \mid k = 1, \ldots, n_s, j = 1, \ldots, J\};
\]
Model for Sources

- Each source $S_k = \{s_{kj} | j = 1, \ldots, J\}$ is an iid Gaussian mixture with an unknown number $m_k$ of components.
- Define $\mu_k = (\mu_{k1}, \ldots, \mu_{km_k})$, $t_k = (t_{k1}, \ldots, t_{km_k})$ and $p_k = (p_{k1}, \ldots, p_{km_k})$ to be the mixture component means, precisions and weights for source $k$;
- So

$$p(S_k | \mu_k, t_k, p_k) = \prod_{j=1}^{J} \sum_{a=1}^{m_k} p_{ka} \sqrt{\frac{t_{ka}}{2\pi}} \exp \left( -0.5 t_{ka} (s_{kj} - \mu_{ka})^2 \right), s_{kj} \in \mathbb{R}.$$ 

- Let $\underline{\mu} = (\mu_1, \ldots, \mu_{ns})$, $\underline{t} = (t_1, \ldots, t_{ns})$, $\underline{p} = (p_1, \ldots, p_{ns})$ and $\underline{m} = (m_1, \ldots, m_{ns})$ denote the vectors of all mixture means, precisions, weights and no. of components for all sources.
Model for Mixing Matrix $A$

- Both $A$ and $s$ unknown $\Rightarrow$ solution up to a constant in each column (source) of $A$;
- Hence can arbitrarily fix one value in each column of $A$;
- $A_{ik}$ interpreted as the response of the detector at frequency $\nu_i$ to source $k$;
- The physics tells us a lot about what this should be for each source;
- The CMB is black body radiation at $T_0 = 2.725K$, so response at $\nu_i$ is

$$A_{i1} = \left(\frac{h\nu_i}{kT_0}\right)^2 \frac{e^{h\nu_i/kT_0}}{(e^{h\nu_i/kT_0} - 1)^2},$$

$h$ is Planck constant, $k$ is Boltzmann’s constant.
Model for Mixing Matrix $A$

- For other sources, physical argument to say that *approximately* we can say:

\[ A_{ik} = \left( \frac{\nu_i}{\nu_{0,k}} \right)^{\theta_k} \]

for a reference frequency $\nu_{0,k}$ and parameter $\theta_k$;

- So one free parameter $\theta_k$ per column of $A$;

- So $A$ parameterised by $(n_s - 1)$ dimensional $\theta$;
Priors

• We’ve parameterised the model in terms of:
  • Source mixture means, precisions, weights and no. of components: \( \mu, t, p, m \);
  • Mixing matrix parameters \( \theta \);
  • Measurement noise precisions \( \tau \);
• We put the usual conjugate priors on these:
  • Normals on the mixture means;
  • Gammas on the mixture precisions;
  • Dirichlets on the mixture weights;
  • Geometrics on the no. of mixture components;
  • Gammas on the noise precisions;
  • For \( \theta \), physical arguments put quite tight bounds on their values — we put normal priors with high probability between these bounds;
• Our existing knowledge can put very informative priors on the source mixture parameters, and on the noise precisions;
• These should greatly help the inference.
Sampling from the Posterior Distribution

- Can be done by Gibbs sampling;
- Update parameters in blocks where possible:
  - Mixture means, precisions and weights updated jointly from their full conditional for each source by a Gibbs sampler;
  - No. of mixture components for each source sampled by the usual Richardson and Green (JRSS B, 1997) reversible jump move;
  - Components of $\theta$ updated jointly with their corresponding source by a Metropolis move (e.g. $(\theta_k, S_k)$);
  - Full conditional of each source at each pixel $s_{kj}$ is a mixture of Gaussians;
    - Better: full conditional of vector of sources at each pixel $S_j$ is a multivariate mixture of Gaussians;
  - Full conditional of noise precisions are gamma;
- See pending paper for the details!!
Example 1: simulated data

- Three sources (simulated Gaussian mixtures and Gaussian MRFs) at five channels on a $256 \times 256$ grid;
- Mixing matrix $A$ generated using reasonable values from CMB, synchrotron and dust at the 5 COBE frequencies, giving:

$$ A = \begin{pmatrix} 0.9770 & 32.8359 & 0.0990 \\ 0.9514 & 10.8140 & 0.2090 \\ 0.8823 & 2.8133 & 0.5107 \\ 0.7770 & 1.0000 & 1.0000 \\ 0.6044 & 0.3544 & 1.9256 \end{pmatrix}.$$
Example 1: simulated data

Figure: Simulated values of the 3 sources, from left to right, assigned to be CMB, synchrotron and dust.
Example 1: simulated data

Figure: Histograms of the simulated values of the 3 sources, from left to right: CMB, synchrotron and dust.
Example 1: simulated data

Figure: Resulting observed signal at two frequencies: 30 GHz (left) and 143 GHz (right).
Example 1: simulated data

Figure: The posterior mean reconstruction of the CMB (left), the true (centre) with a scatter plot of true vs posterior mean (right).
Example 1: simulated data

**Figure:** On the left, the posterior distribution of the CMB at pixel (200,20). The true value is indicated by the vertical line. On the right, the marginal posterior distribution of the CMB, with the histogram of true values for comparison.
Example 1: simulated data

**Figure:** The posterior mean reconstruction of synchrotron (left), the true (centre) with a scatter plot of true vs posterior mean (right).
Example 1: simulated data

Figure: The posterior mean reconstruction of dust (left), the true (centre) with a scatter plot of true vs posterior mean (right).
**Example 1: simulated data**

**Figure:** The fitted marginal posterior distribution of synchrotron (left) and dust (right), along with the histogram of their true values for comparison.
Example 2: simulated data with 9 channels

- Same 5 channels as example 1 with 4 more at higher frequencies;

\[
A = \begin{pmatrix}
0.9770 & 32.8359 & 0.0990 \\
0.9514 & 10.8140 & 0.2090 \\
0.8823 & 2.8133 & 0.5107 \\
0.7770 & 1.0000 & 1.0000 \\
0.6044 & 0.3544 & 1.9256 \\
0.2194 & 0.1057 & 5.3763 \\
0.0294 & 0.0258 & 11.3455 \\
0.0019 & 0.0073 & 17.5890 \\
0.0001 & 0.0020 & 21.9472
\end{pmatrix},
\]

- Higher frequencies give much higher to dust.
Example 2: simulated data with 9 channels

Figure: The posterior mean reconstruction of the CMB (left), the true (centre) with a scatter plot of true vs posterior mean (right).
Example 2: simulated data with 9 channels

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Real WMAP Data

- Three patches of $512 \times 512$ pixels at 5 channels;
- Fit 4 sources: CMB, synchrotron, dust and free-free emission;
- The spectral index of free-free emission is assumed to be $-2.19$. 
Patch 1 — Data
Patch 1 — Posterior mean of sources
Patch 1 — Posterior standard deviation of sources
Patch 1 — Model fit: observed temperature vs. posterior mean temperature
Patch 2 — Data
Patch 2 — Posterior mean of sources
Patch 2 — Posterior standard deviation of sources
Patch 2 — Model fit: observed temperature vs. posterior mean temperature
Patch 3 — Data
Patch 3 — Posterior mean of sources

![Posterior mean CMB](image1)
![Posterior mean synchrotron](image2)

![Posterior mean dust](image3)
![Posterior mean free-free](image4)
Patch 3 — Posterior standard deviation of sources
Patch 3 — Model fit: observed temperature vs. posterior mean temperature
Dust spectral index varies considerably from patch to patch!!

Note: free-free spectral index assumed to be $-2.19$ (as in Eriksen et al., 2006).

<table>
<thead>
<tr>
<th>Patch</th>
<th>Synchrotron</th>
<th>Dust</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-2.61$</td>
<td>3.40</td>
</tr>
<tr>
<td>2</td>
<td>$-2.84$</td>
<td>1.34</td>
</tr>
<tr>
<td>3</td>
<td>$-2.64$</td>
<td>0.51</td>
</tr>
</tbody>
</table>
Future Work

- Dependent sources (CMB vs. galactic vs. extra-galactic) by mixtures of multivariate Gaussians;
- Analysis of entire WMAP data;
- Separation at resolution of shortest wavelength;
- Planck data from 2008.