Wedgelets Partitions and Image Processing

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1. Geometry in Images: Paradigms

2. Wedgelet Segmentations

3. Data structure and compression

4. Illustrations
Classical Compression Standards

Original Image

JPEG (6.8 KB)
DCT

JPEG2000 (6.5 KB)
FWT + Contexts
Zoom

Original Image

JPEG (6.8 KB)  
DCT

JPEG2000 (6.5 KB)  
FWT + Contexts
Mathematical Background

**Justification** Approximation Theory

Image: real-valued function, continuous domain

**Ansatz** natural images have some regularity

\[ f \in X \text{ (quasi?)-Banach space}, \; X \subset \subset L^2(\Omega) \]

**Approximation** \( \hat{f}_n = \sum_i \alpha_i \varphi_i, \varphi \in A, \; n\)-approximation

We look for \( A \), such that

\[ \| f - \hat{f}_n \|_2^2 = O \left( \frac{1}{n^\alpha} \right), \text{ for some } \alpha > 0, \text{ and } f \in X \]

**Critics**

- Asymptotic results
- Continuous vs Discrete
Old and new Ansätze

Orthogonal Transforms

► FOURIER: non optimal (bad for local singularities)

► WAVELETS: optimal Non Linear Approximation rates for Besov spaces and Bounded Variation

+ in 2D Isotropic vs Anisotropic Methods

⇒ Structure of the set of singularities

Geometrical Methods

► CURVELETS (Candès 99-04) + SHEARLETS (Labate et al. 2005) not adaptive but quasi-optimal (flexible geometrical features)

► BANDELETS (Mallat-LePennec 99)

► TRIANGULATIONS: good theoretical Approximation rates (Mallat 04, Demaret-Iske 06)

► WEDGELETS (Donoho 99)
DIAGRAM OF FUNCTIONAL SPACES
(IN 2D)

\( \ell^q([0,1]^d) \)

\( \ell^0([0,1]^d) \)

\( L^\infty([0,1]^d) \)

\( L^2([0,1]^d) \)

\( L^4([0,1]^d) \)

\( \frac{1}{\rho} \)

NL Approx.
Wavelets and Contours

Wedge (left) and its Wavelet coefficients (right)
Geometrical Segmentations

\( S \subset \mathbb{Z}^2 \) \hspace{1cm} \text{set of pixels}

\( f \in \mathbb{R}^S \) \hspace{1cm} \text{image}

\( \mathcal{P} \) \hspace{1cm} \text{family of partitions} \hspace{0.3cm} \mathcal{P} \subset 2^S \text{ of } S

\( f_\mathcal{P} \in \mathbb{R}^S \) \hspace{1cm} \text{best constant approximation with} \hspace{0.3cm} f_\mathcal{P}|_r \text{ constant, } r \in \mathcal{P}

\( \mathcal{G} \) \hspace{1cm} \text{segmentations} \hspace{0.3cm} (\mathcal{P}, f_\mathcal{P})

\( \gamma \geq 0 \) \hspace{1cm} \text{penalisation parameter}

**Goal:** Efficient Minimisation of the penalised Functional

\[
H_{f,\gamma} : \mathcal{G} \rightarrow \mathbb{R}, \quad (\mathcal{P}, f_\mathcal{P}) \mapsto \gamma \cdot |\mathcal{P}| + \|f - f_\mathcal{P}\|_2^2 \quad (\gamma \geq 0).
\]

**Result**

\[
(\hat{\mathcal{P}}, \hat{f}_\mathcal{P}) \in \arg\min_{(\mathcal{P}, f_\mathcal{P})} H_{z,\gamma}
\]

optimal tradeoff between penalisation and reconstruction quality
Wedgelet Segmentations

\[ H_{f,\gamma} : \mathcal{G} \rightarrow \mathbb{R}, \quad (P, f_P) \mapsto \gamma \cdot |P| + \|f - f_P\|_2^2 \quad (\gamma \geq 0). \]

- **Problem** Size of the search space :\(|\mathcal{P}| > 2^{|\mathcal{S}|!}\)
  - MCMC: slow and not exact
- Restriction of the search space
  - discrete wedges
  - nested Quadtree structure
- fast moment computation: Green-like formula
Representation Elements

- DCT basis (JPEG)
- (Haar) Wavelet basis
- Wedgelet partitions
Data Structure

Quadtree Partition
dyadic Wedge Partition
Example
Idee  Wedgelet representation contains too much redundancies
⇒ Correlation Model between neighbours

ALGORITHM

- Tree Coding
- Model Coding
- IF (Model = constant over square)
  (quantised) mean value encoded
- IF (Modell = constant over each Wedge)
  Angle Encoding and relative position Coding of the (quantised) mean values
Compression: Features

- mixed Models (e.g. square constant, wedge constant, wedge linear ...)
- corresponding penalisation: estimation of the coding costs

\[ H_\gamma : (f, (P, f_P)) \mapsto \gamma \left( \sum_i |C(W_i)| + \sum_j |C(Q_j)| \right) + \|f - f_P\|_2^2, \quad \gamma \geq 0, \]

- \( W_i \): wedge, \( Q_j \): square, \( C \): estimator for the coding costs
- Coding
  - combinatorial encoding
  - angle coding: resolution-adaptive
- Prediction Method
Prediction

- **Observation** Representation still strongly redundant
  - "not natural", arbitrary quadtree structure

- **Main idea**
  - Multiresolution differential coding
    - only "Brotherhood" correlations
  - Extraction of spatial correlation between quadtree "cousins"

- Current piece coded from the causal (already coded) information
How to Code the Leaves?

Tree

Levels of the leaves
Predictive Coding: an Illustration

- **Binary Tree**: 45 bits
- **Bottom to Top Non-Predictive**: \( \log_2(64) + \log_2(\binom{64}{8}) = 39 \) bits
- **Bottom to Top Predictive**: \( \log_2(64) + \log_2(\binom{42}{1}) + \log_2(\binom{22}{7}) = 31 \) bits
First results (1)

Comparison between "pure Wedge" and "Wedge+Constant" Models with higher penalisation for Wedges versus Squares

\[ C(W_i) = 3.5 \times C(Q_i) \]

(a) Original Image (b) Squares: 87784 b, 30,54 dB (c) only Wedges: 84632 b, 30,42 dB (d) Wedges + Squares: 76184 b, 30,60 dB

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First Results (2)

Circles, WC, 533 B, PSNR: 27.50 dB

Peppers, WC, 10.5 KB, PSNR: 31.50 dB
Compression Rate 1:25
systematic investigation of the penalisation functional
  - rate-distortion Optimisation
  - Depends on the resolution
  - Contexts change penalty

Contextual Encoding

Compression with richer regression models (e.g. linear)
  - aim: avoid bloc artefacts

Correct theoretical framework for discrete Data
  - non asymptotical results