Connections between random Boolean networks and their annealed model

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Outline

1. Introduction
   - Boolean networks
   - Random Boolean networks
   - Annealed and quenched model

2. Main Part
   - Sensitivity of Boolean functions
   - Analysis of the quenched model
   - Analysis of the annealed model

3. Summary
A Boolean network consists of $N$ interconnected nodes $i$ each capable of storing a binary value.

Each node $i$ has $K$ input edges $j_{i1}, \ldots, j_{iK}$.

To each node a Boolean function $f_i$ is assigned.
Define $s_i(t)$ as the value stored by $i$ at time $t$.

Then:

$$s_i(t + 1) = f(s_i_1(t), \ldots, s_i_K(t))$$
NK networks: Random Boolean networks with $N$ nodes, where:

- for each node a Boolean function is chosen among all equally likely functions with $K$ arguments,
- for each function the $K$ arguments are chosen among $\binom{N}{K}$ equally likely possibilities,
- finally a random initial state is chosen.

By numerical simulations S. Kauffman found that if $K \leq 2$, the networks show ordered behaviour:

- Large proportion of weak nodes.
- Large proportion of frozen nodes.
- Small attractor cycles.

Contrary if $K > 2$ the networks are disordered or chaotic.
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Contrary if $K > 2$ the networks are disordered or chaotic.
Here we consider the ensemble RBN\((K, P)\):

- for each node a Boolean function \(f\) with \(K\) arguments is chosen as follows: each of the \(2^K\) positions in the truth table of \(f\) is set to 1 with probability \(P\).
- for each function the \(K\) arguments are chosen among \({N \choose K}\) equally likely possibilities,
- finally a random initial state is chosen.
[Derrida & Pomeau 1986] introduced the *annealed model*. In contrast to the classical model (the so called *quenched model*) the functions and connections are chosen at random at each time step.
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**Theorem**

*Ordered behaviour, which means*

\[
\frac{\mathbb{E}(d_H(s_1(t), s_2(t)))}{N} \to 0,
\]

*if and only if*

\[
2KP(1 - P) \leq 1.
\]
Consider a network (with $N$ nodes) with an arbitrary state.

- A node $G$ is $t$-weak if a perturbation of $G$ vanishes in $t$ steps.
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**Theorem**

*Ordered behaviour, which means*

\[
\lim_{N \to \infty} \Pr(G \text{ is } \alpha \log N\text{-weak}) = 1,
\]

*if and only if*

\[
2KP(1 - P) \leq 1.
\]

($\alpha$ is a constant depending on $K$ only)
Motivation

Question

- What is the connection between the two models?
Definition

- The \( l \)-sensitivity \( s^l_f(w) \) of a function \( f \) with argument \( w \in \mathbb{F}^K_2 \) is the number of vectors \( x \) in Hamming distance \( l \) to \( w \), for which \( f(w) \neq f(x) \).
- The average \( l \)-sensitivity \( s^l_f \) is the average of \( s^l_f(w) \) of all \( w \).
**Definition**

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- The *average l*-sensitivity $s^l_f$ is the average of $s^l_f(w)$ of all $w$.

**Example:**

$$f((w_1, w_2, w_3)) = w_1 \oplus w_2 \oplus w_3$$

Argument space of $f$, $f = 1$ marked red:
**Definition**

- The $l$-sensitivity $s^l_f(w)$ of a function $f$ with argument $w \in \mathbb{F}_2^K$ is the number of vectors $x$ in Hamming distance $l$ to $w$, for which $f(w) \neq f(x)$.
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**Example:**

$$f((w_1, w_2, w_3)) = w_1 \oplus w_2 \oplus w_3$$

for all $w$:

$$s^1_f(w) = 3 \quad \text{and} \quad s^2_f(w) = 0$$
Suppose that Boolean functions are chosen at random. The probability of choosing a function $f$ is given by $p_f$: The expectation of the $l$-sensitivity is given by

$$E\left(s^l_f(w)\right) = \sum_f p_f s^l_f(w).$$
Expectation of $l$-Sensitivity for random function

Suppose that Boolean functions are chosen at random. The probability of choosing a function $f$ is given by $p_f$: The expectation of the $l$-sensitivity is given by

$$\mathbb{E} \left( s^l_f(w) \right) = \sum_f p_f s^l_f(w).$$

Similar

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$$

For RBN($K, P$) it turns out that

**Lemma**

for all $w$:

$$
\mathbb{E}\left(s^l_f(w)\right) = \text{const.} = \mathbb{E}\left(s^l_f\right).
$$
Suppose that Boolean functions are chosen at random. The probability of choosing a function $f$ is given by $p_f$: The expectation of the $l$-sensitivity is given by

$$\mathbb{E}\left(s_f^l(w)\right) = \sum_f p_f s_f^l(w).$$

Similar

$$\mathbb{E}\left(s_f^l\right) = \sum_f p_f s_f^l.$$

For RBN($K, P$) it turns out that

**Lemma**

$$\mathbb{E}\left(s_f^l\right) = \frac{{K \choose l}}{K} \mathbb{E}(s_f^1).$$
Lynchs order parameter

Suppose
- the probability for a function $f$ is given by $p_f$ and
- the *mean activity* is independent of time and given by $a$. 

\[ \lambda = \sum f \cdot p_f \sum w \in F K_2^s f(w) a \cdot H(w) (1 - a) K_i - w H(w), \]

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Suppose

- the probability for a function \( f \) is given by \( p_f \) and
- the *mean activity* is independent of time and given by \( a \).

**Definition (Lynch)**

\[
\lambda = \sum_f p_f \sum_{w \in \mathbb{F}_2^K} s_f(w) a^{w_H(w)} (1 - a)^{K_i - w_H(w)},
\]
Suppose
- the probability for a function $f$ is given by $p_f$ and
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**Definition (Lynch)**

$$\lambda = \sum_f p_f \sum_{w \in F_2^K} s_f(w) a^{w_H(w)} (1 - a)^{K_i - w_H(w)}.$$ 

in general

**Theorem (Lynch)**

*Ordered behaviour if and only if*

$$\lambda \leq 1.$$
Suppose
- the probability for a function $f$ is given by $p_f$ and
- the mean activity is independent of time and given by $a$.

**Definition (Lynch)**

$$\lambda = \sum_{f} p_f \sum_{w \in \mathbb{F}_2^K} s_f(w) a^{w_H(w)} (1 - a)^{K_i - w_H(w)},$$

for $\text{RBN}(K, p)$:

**Theorem**

$$\lambda = \mathbb{E}(s_f^1)$$

*hence ordered behaviour if and only if*

$$\mathbb{E}(s_f^1) \leq 1.$$
Consider two instances of the same random network (with $N$ nodes) starting from two different initial states ($s_1, s_2$). Define the fractional overlap $a(t) = 1 - \frac{\mathbb{E}(d_H(s_1(t), s_2(t)))}{N}$. 
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At time $t$: Define a set of nodes $A_t$ which store the same value in both instances (marked as yellow below).

Yellow nodes: same value in both instances
Red nodes: different values in both instances
Consider two instances of the same random network (with $N$ nodes) starting from two different initial states ($s_1, s_2$). Define the fractional overlap $a(t) = 1 - \frac{\mathbb{E}(d_H(s_1(t), s_2(t)))}{N}$.

At time $t$: Define a set of nodes $A_t$ which store the same value in both instances (marked as yellow below).

Next time step: there are nodes (blue) that receive their input only from $A_t$. We expect $Na(t)^K$ blue nodes and $N(1 - a(t)^K)$ other nodes at time $t + 1$, the latter having probability $P_d$ of being different. Therefore:

$$a(t + 1) = a(t)^K + (1 - P_d)(1 - a(t)^K).$$
Suppose that \( s_1 \) and \( s_2 \) are randomly chosen but different and \( f \) is a random function.

\[
P_d = Pr ( f(s_1) \neq f(s_2) \mid s_1 \neq s_2 )
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For \( \text{RBN}(K, P) \) it turns out that

\[
P_d = \frac{\mathbb{E}(s^1_f)}{K}.
\]
Therefore $a(t)$ evolves according a one-dimensional map

$$a(t + 1) = A(a(t))$$

where

$$A(x) = 1 + P_d(x^K - 1) = 1 + \mathbb{E}(s_f^1)(x^K - 1).$$
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\]

**Theorem**

*Stable fixed point* \( x_0 = 1 \) (*total overlap, hence ordered behaviour*) if and only if

\[
\mathbb{E}(s^1_f) \leq 1.
\]
Summery and comments

Due to the simple form of the expectation of the $l$-sensitivity:

- For $\text{RBN}(K, P)$ the phase of both models, the annealed and the quenched, is determined by the expectation of the average sensitivity (order 1).
- This is also true for other ensembles (see paper).
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- For $\text{RBN}(K, P)$ the phase of both models, the annealed and the quenched, is determined by the expectation of the average sensitivity (order 1).
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Note:
It can be shown, that similar results hold, if the probability of a function is only dependent on the number of 1 in the truth table (not yet published).
Thank you

for your attention!