Are there intuitive relaxations of planarity that support a lower bound on the independence ratio?

Sometimes the independence ratio is more fun to look at than the chromatic number.
Outline

1. Introduction: the independence ratio
2. Embedded graphs
   a) Early results
   b) Open questions
3. Graphs with given thickness
4. Graphs with given crossing number
5. Independence questions for new varieties of nearly planar graphs
The Independence Ratio

The Fraction [V65]; The Name [AH75]

Suppose $G$ is a graph with $n$ vertices. Let

$$\alpha(G) = \max\{|U| : U \subseteq V(G); \ x, y \in U \Rightarrow xy \notin E(G)\}.$$  

The \textit{independence ratio}, \(\mu(G)\), is defined by

$$\mu(G) = \frac{\alpha(G)}{n}.$$  

Since a color class is independent, \(\alpha(G) \geq \frac{n}{\chi(G)}\).

Thus \(\mu(G) \geq \frac{1}{\chi(G)}\).

There is a circular refinement \textit{viz.} \(\mu(G) \geq \frac{1}{\chi_c(G)}\).
Planar Graphs

**Conj** [EV 60s] If $G$ is planar, then $\mu(G) \geq \frac{1}{4}$.
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![Graph](image.png)
Planar Graphs

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EV is sharp

Th [A74] If $G$ is planar, then $\mu(G) > \frac{2}{9}$.

4CT $\Rightarrow$ EV \{still no independent proof\}
Embedded Graphs (we know a lot)

Let $S_g$ denote the orientable surface with $g$ handles.

**Th** [H91] If $G$ is embedded on $S_g$, then
\[ \chi(G) \leq H(g) = \left\lfloor \frac{7 + \sqrt{48g + 1}}{2} \right\rfloor. \]
Thus $\mu(G) \geq \frac{1}{H(g)}$.

**Cor** If $G$ is toroidal, then $\mu(G) \geq \frac{1}{7}$.

**Th** [RY68] $K_{H(g)}$ embeds on $S_g$. 

Th [AH75] Suppose $G \neq K_7, K_6, K_7 \cup K_4$, or $C_{11}^3$. If $G$ is toroidal, then $\mu(G) \geq \frac{1}{5}$.
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**Th [AH75]** Suppose \( G \neq K_7, K_6, K_7 \cup K_4, \text{or} \ C_1^{11} \). If \( G \) is toroidal, then \( \mu(G) \geq \frac{1}{5} \).

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Th [AH78, S] Suppose $G$ embeds on $S$. Given $\epsilon > 0$, $\exists N(\epsilon, S) : \text{if } n > N(\epsilon, S), \text{then } \mu(G) > \frac{1}{4} - \epsilon$. 

11
On any given $S$ only a few graphs have $\mu << \frac{1}{4}$.

**Sketch of Proof Technique:** Cycle $C \subset G$ embedded on $S$, is *n.c.* if it is not homotopic to a point.

**Width of $G$** [AH75] $w(G) = \min\{|C| : C \text{ is n.c.}\}$.
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**Width of $G$** [AH75] $w(G) = \min\{|C| : C \text{ is n.c.}\}$.

**Th** [AH78] If $G$ triangulates $S$, then $w(G) \leq \sqrt{2n}$.

**Cor** If $G$ is embedded on $S$, then $\exists \, U \subset V(G) : |U|$ is small and $G[V - U]$ is planar.
Questions for Embedded Graphs [AH74]

Background:

**Th** [AS82] $G$ toroidal and $w(G) > 3 \Rightarrow \chi(G) \leq 5$.

**Cor** If $G$ is toroidal, then $\mu(G) \geq \frac{1}{5} - \frac{3}{5n}$.
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**Q** Does $M_{Sg} = 3g$?
So where are we?

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Embedding similar to planarity wrt “$\mu$”-behavior.
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Where are we going?

What happens with other relaxations of planarity?

If $G$ is nearly planar, what about $\mu(G)$?
Nearly Planar Graphs

• Classic Versions
  - thickness
  - crossing number

• Recent versions
  - locally planar graphs
  - $k$-quasi-planar graphs
  - $k$-embedded graphs
  - $k$-quasi*planar graphs
**Thickness** (we don’t know much)

$G$ is said to have thickness $t$ if $G$ is the union of $t$ planar graphs but no fewer.

**Rmks** If $G$ has thickness $t$, then $E \leq t(3n - 6)$. So, if $G$ has thickness $t$, then $\chi(G) \leq 6t$. 

$\exists G$ with thickness $t$ such that $\chi(G) \geq 6t - 2$ ($t > 2$). When $t = 2$, all we know is $9 \leq \chi(G) \leq 12$.

**Cor** If $t(G) = 2$, then $\mu(G) \geq \frac{1}{12}$.

**Th** [BH,AG] $t(K_{n}) = \left\lfloor \frac{n+7}{6} \right\rfloor$ ($n \neq 9, 10$) 

$t(K_{9}) = t(K_{10}) = 3$
Independence for Thickness 2 Graphs

• $\exists \mu_2 : t(G) = 2 \Rightarrow \mu(G) \geq \mu_2 \geq \frac{1}{12}$.

Old Conj [A] $\mu_2 = \frac{1}{8}$.
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Open Q What is \( \mu_2 \)?

Q [A] Given \( \epsilon > 0 \), \( \exists G : t(G) = 2, \frac{1}{9} < \mu(G) < \frac{1}{9} + \epsilon \)?
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Conj [G] $\mu_2 = \frac{2}{21}$. 
Crossing Number (we know even less)

The crossing number of $G$ ($\text{cr}(G)$) is the minimum number of crossings in any drawing of $G$.

Conj $\text{cr}(K_n) = Z_n = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$.

Th [KMPRS] $\lim_{n \to \infty} \text{cr}(K_n)/Z_n \geq 0.83$.

Q Is $\chi(G)$ bounded by a function of ($\text{cr}(G)$)?

Q Is $\mu(G)$ bounded by a function of ($\text{cr}(G)$)?
Small Results on Crossings and Colorings [OZ]

**Th** If \( \text{cr}(G) \leq 2 \), then \( \chi(G) \leq 5 \).

Notation: \( \omega(G) \) denotes the *clique number*.

**Th** If \( \text{cr}(G) \leq 3 \) and \( \omega(G) \leq 5 \), then \( \chi(G) \leq 5 \).

**Q** If \( \text{cr}(G) \leq 5 \) and \( \omega(G) \leq 5 \), is \( \chi(G) \leq 5 \)?
A Small Result on Crossings and Colorings [A]

Def In a plane graph, two crossings are *dependent* if their eight incident vertices are *not* distinct.

![Diagram of crossings and vertices]

{(av, bu)(vc, bw)} dependent
{(av, bu)(cx, dw)} not dependent

Th If $G$ is a plane graph, $cr(G) \leq 3$, and crossings are independent, then $\chi(G) \leq 5$. Thus $\mu(G) \geq \frac{1}{5}$.

Conj If $G$ is a plane graph and no two crossings are dependent, then $\chi(G) \leq 5$ and $\mu(G) \geq \frac{1}{5}$. 
Rmk [A,S] If $G$ is a plane graph and no two crossings are dependent, then $\chi(G) \leq 8$. Thus $\mu(G) \geq \frac{1}{8}$.

Th [A] If $G$ is a plane graph and no two crossings are dependent, then $\chi(G) \leq 6$. Thus $\mu(G) \geq \frac{1}{6}$.
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**Pf** From crossing independence, $n \geq 4 \cdot \text{cr}(G)$. Thus $\exists U \subset V(G): |U| \leq \frac{n}{4}$ & $G[V - U]$ is planar.

$\alpha(G) \geq \alpha(G[V - U]) \geq \frac{1}{4} \cdot \frac{3n}{4} = \frac{3n}{16}$. 
**Th** [A] If $G$ is a plane graph and no two crossings are dependent, then $\mu(G) \geq \frac{3}{16}$.

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**Rmk** The proof of the $\mu$-result is easier than the proof of the $\chi$-result, but the $\mu$-result is stronger.
A Naive Definition of Locally Planar

Given $x \in V(G)$, let

$$N_d^d[x] = \{u \in V(G) : \text{dist}(x, u) \leq d\}.$$ 

If $G[N_d^d[x]]$ is planar $\forall x \in V(G)$ and $d$ is large, we could say that $G$ seems locally planar. However,

Th [E59] $\forall k, m \in \mathbb{Z}$ there exists a graph $G$ such that $\chi(G) \geq k$, and the girth of $G \geq m$. 
Locally Planar Embedded Graphs

Def Suppose $G$ is embedded on $S$. If $w(G)$ is large, we say that $G$ is \textit{locally planar}.

Note if $d < \frac{w(G)-1}{2}$, then $\forall x, G[N^d[x]]$ is planar.

The previously mentioned results on the independence ratio justify the above definition.

In addition there are similar coloring results.
**Th** [H84] If $G$ is embedded on $S_g$ and every edge is short enough, then $\chi(G) \leq 5$.

**Th** [T93] If $G$ is embedded on $S_g$ and $w(G) \geq 2^{28g+6}$, then $\chi(G) \leq 5$.

**Th** [DKM05] If $G$ is embedded on $S_g$ and $w(G)$ is large enough, then $\chi_\ell(G) \leq 5$. 
A Question on Local Planarity and Thickness

Suppose $G$ is a graph with thickness 2.

For $1 \leq r \leq 4$, does there exist $d = d(r)$:

if $G[N^d[x]]$ is planar, then $\mu(G) \geq \frac{1}{4+r}$?
New Nearly Planar Graphs

Here are recent attempts to capture near planarity.

Some come with extremal results about $|E(G)|$.

For each attempt: Is there an idea to get from the intuitively attractive definition to a meaningful theorem about $\mu$?
Another Version of Locally Planar

**Def** [PPTT02] $G$ is said to be $r$-locally planar if $G$ contains no self intersecting path of length $\leq r$.

**Th** [PPTT02] $\exists$ 3-locally planar graphs with $E \geq c \cdot n \log(n)$.

**Th** [PPTT04] If $G$ is 3-locally planar, then $E = O(n \log(n))$. 
\[ \exists \mu_3 \text{ such that every } 3\text{-locally planar graph has } \mu(G) \geq \mu_3? \]
Q  \exists \mu_3 \text{ such that every 3-locally planar graph has } \mu(G) \geq \mu_3 ?

These graphs can have a superlinear number of edges \( \Rightarrow \) greedily building an independent set does not work.
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Looking at $r$-locally planar graphs ($r \geq 4$) gives similar results.
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These graphs can have a superlinear number of edges \Rightarrow \text{ greedily building an independent set does not work.} \\

Looking at } r\text{-locally planar graphs } (r \geq 4) \text{ gives similar results.} \\

The examples of } r\text{-locally planar graphs with lots of edges have relatively large } \mu.
Quasi Planar Graphs

**Def [PT97]** If, $G$ has a drawing in which no edge crosses more than $r$ other edges, we say that $G$ is r-quasi planar (r-q-p).

**Th [PT97]** If $G$ is r-q-p, then $E \leq (r + 3)(n - 2)$. Sharp for $0 \leq r \leq 2$ - not close for large $r$.

**Cor** If $G$ is r-q-p, then $\mu(G) \geq \frac{1}{2r+6}$.
\textbf{Th} [B84] If $G$ is 1-q-p, then $\chi(G) \leq 6 \Rightarrow \mu(G) \geq \frac{1}{6}$.

The above theorem is sharp
Def [R] Given a planar graph $G$, the vertex-face graph $G_{vf}$ has $V(G_{vf}) = V(G) \cup F(G)$. 

$E(G_{vf}) = \{xy : x$ is adjacent to or incident with $y\}$.
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**Rmks** $G$ planar $\Rightarrow G_{vf}$ is 1-q-p. $K_6 \not\cong G_{vf}$.

**Conj** [A] If $G$ is planar, then $\mu(G_{vf}) \geq \frac{2}{11}$.

$\mu((K_3 \square K_2)_{vf}) = \frac{2}{11}$.
$K_3 \square K_2$
$K_3 \Box K_2$

$(K_3 \Box K_2)_D$
$K_3 \square K_2$

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**Th** [R, AM06] If $G$ is 1-embeddable on $S_g$, then

$$
\chi_\ell(G) \leq \left\lfloor \frac{9 + \sqrt{32g + 17}}{2} \right\rfloor = R(g) \Rightarrow \mu(G) \geq \frac{1}{R(g)}.
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Th [AM06] If $G$ is 1-embeddable on $S_g$ and $w(G) \geq 104g - 204$, then $\chi_\ell(G) \leq 8 \Rightarrow \mu(G) \geq \frac{1}{8}$.
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Q If $G$ is 1-embedded on $S_g$ and $w(G)$ is large enough, is $\mu(G) \geq \frac{1}{6}$?
An Alternate Definition of Q-P

**Def** [AAPPS95] A graph is *k*-quasi*planar* if it has a drawing in which no *k* of its edges are pairwise crossing.

A 3-quasi*planar* graph
Th [AAPPS95] If $G$ is 3-quasi*planar, then $E = O(n)$. 
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**Th** [AcT05] If $G$ is simple and 3-quasi*planar, then $E < 6.5n$ The bound is sharp except for a subtractive constant.
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Th [AAPPS95] If $G$ is 3-quasi*planar, then $E = O(n)$.

Th [AcT05] If $G$ is simple and 3-quasi*planar, then $E < 6.5n$. The bound is sharp except for a subtractive constant.

Cor If $G$ is 3-quasi*planar then $\mu(G) \geq \frac{1}{13}$.

The graphs which show that the edge bound is sharp have $\mu \geq \frac{1}{6}$. 
Th [Ac05] If $G$ is 4-quasi*planar, then $E \leq 36(n - 2)$.

Q Do $k$-quasi*planar graphs have a linear number of edges?

Q If $G$ is $k$-quasi*planar (especially when $k = 3$), what is the best bound for $\mu(G)$?