Inference for dynamics of continuous variables: the Extended Plefka Expansion with hidden nodes

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**Introduction and Motivation**

**Fundamental and practical limitations:**

- Non-linear equations
- Vast amount of information ($\sim 10^4$ eqs)
- **Uncertainty** → No complete qualitative understanding

**SIZE, COMPLEXITY, UNCERTAINTY**

↓

Statistical Physics for Model Reduction

EGFR network from *Kholodenko et al.* (1999)
Let us assume it is possible to characterize **only some** nodes

Small subset of variables: **Subnetwork** → “observed”
Embedded in a larger network: **Bulk** → “hidden” (unknown)

**Question**

What can we say in general about the **INFERENCE** of hidden dynamics?
LINEAR DYNAMICS

\[ i, j \rightarrow \text{hidden (Bulk)} \]
\[ a, b \rightarrow \text{observed (Subnetwork)} \]
\[ \xi_i, \xi_a \text{ Gaussian white noises} \]

\[ \langle \xi_i(t)\xi_j(t') \rangle = \sigma_b^2 \delta_{ij} \delta(t - t') \]
\[ \langle \xi_a(t)\xi_b(t') \rangle = \sigma_s^2 \delta_{ab} \delta(t - t') \]

\[ \partial_t x_i(t) = -\lambda x_i(t) + \sum_j J_{ij} x_j(t) + \sum_a K_{ia} x_a(t) + \xi_i(t) \]

\[ \partial_t x_a(t) = -\lambda x_a(t) + \sum_b J_{ab} x_b(t) + \sum_j K_{aj} x_j(t) + \xi_a(t) \]
DYNAMICAL MEAN FIELD APPROXIMATION

1. Effectively **non-interacting** dynamics:
   Couplings are replaced by **memory + coloured** noise

2. Extension: 1\textsuperscript{st} and 2\textsuperscript{nd} moments are constrained

3. **Gaussian** statistics conditioned on observations

**Posterior mean** $\mu_i(t) =$ best estimate of hidden dynamics

**Forward-Backward** propagation

**Posterior variance** $C_i(t, t) =$ estimate of **error** of prediction
Expect Plefka to give exact $\mu_i(t)$ and $C_i(t, t)$ (=errors) when 
Mean Field interactions + Thermodynamic limit $N^B \to \infty$

**Stationary Regime** $\to$ Time Translation Invariant 
Average $\tilde{C}^B(\omega) = \frac{1}{N^B} \sum_i \tilde{C}_i(\omega)$ in **Fourier** space

$$\tilde{C}^B(\omega) = \frac{\sigma_s^2}{k^2} \underbrace{C^B_{\alpha, \gamma, \eta, p}(\Omega)}_{\text{Amplitude}}$$

**Dimensionless parameters**

- $\alpha = \frac{N^S}{N^B}$ Ratio observations/hidden states
- $\gamma = \frac{i}{\lambda}$ Stability of hidden dynamics $\gamma < \gamma_c = \frac{1}{1+\eta}$
- $p = \frac{\lambda}{\sigma}$ Decay constant & hidden-to-observed coupling

$\sigma = \frac{\sigma_b k}{\sigma_s}$ Defines the frequency scale: $\Omega = \frac{\omega}{\sigma}$
We analyze in the parameter space $\alpha, \gamma, p$ the **singularities** of $C^B(0)$

**Critical regions**

1. $\forall p$, $\alpha = 0$ and $\gamma > \gamma_c$
   
   No observations: internal stability

2. $\forall \gamma$, $p = 0$ and $0 < \alpha < 1$
   
   (i.e. $k \gg \lambda$ at fixed $\sigma_s / \sigma_b$)
   
   “Underconstrained” hidden system: strong constraints from observations, but too few
Power-law dependence of $C^B(0)$ on $\delta\alpha$, $\delta\gamma$, $p$:

**Scaling analysis ⇒ Master curves**

Information on relaxation times and inference error

$\Omega^* \ll \Omega \ll 1 \quad \alpha \to 0 \quad C^B \sim \frac{1}{\Omega^2} \quad \alpha \to 1 \quad C^B \sim \frac{1}{\Omega}$

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Inference for dynamics
Thank you for your attention!