Goal Directed Tracing of Inferences in OWL EL Ontologies

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Summary of this talk

A new technique for inference tracing in OWL EL
(a lightweight language for large terminologies):

- finding all inference steps used in a proof of a particular class subsumption ($A \sqsubseteq B$)
- doing it fast

The method can be used with arbitrary rule-based procedures but the main ideas are demonstrated for $\mathcal{EL}$ reasoning

Implemented ELK 0.5.0+, works well in practice

Consequence-based reasoning in $\mathcal{EL}$: example

$\mathcal{O} = \{\mathsf{A} \sqsubseteq \exists \mathsf{R}.\mathsf{B},\quad \mathsf{B} \sqsubseteq \mathsf{D},\quad \mathsf{D} \sqsubseteq \exists \mathsf{S}.\mathsf{A},\quad \exists \mathsf{R}.\mathsf{D} \sqsubseteq \mathsf{C},\quad \exists \mathsf{S}.\mathsf{C} \sqsubseteq \mathsf{C}\ \}$

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$$\overset{\mathbf{R_0}}{\longmapsto} A \sqsubseteq A$$

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$A \sqsubseteq A \qquad A \sqsubseteq \exists R.B$

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Explain: how was $B \sqsubseteq C$ derived?

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Explain: how was $B \sqsubseteq C$ derived?

Closure under the $\mathcal{EL}$ rules contains

- enough information on what is derived
- no information on how or why it is derived

Explanations are important (e.g. for debugging), hard and require tool support

Approaches to explanations

Justifications: minimal subsets of $\mathcal{O}$ which entail the axiom

- don't provide information on how the result follows
- intractable even for very simple languages (in the worst case)
  often work fine in practice

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Proof-based explanations present steps made by the reasoning procedure to derive the result

$$\xmapsto{\mathbf{R_0}} A \sqsubseteq A \xmapsto{\mathbf{R}_\sqsubseteq} A \sqsubseteq \exists R.B \xmapsto{\mathbf{R}_\exists} A \sqsubseteq \exists R.D \xmapsto{\mathbf{R}_\sqsubseteq} A \sqsubseteq C$$

$$\xmapsto[\mathbf{R_0}]{} B \sqsubseteq B \xmapsto[\mathbf{R}_\sqsubseteq]{} B \sqsubseteq D \xmapsto[\mathbf{R}_\sqsubseteq]{} B \sqsubseteq \exists S.A \xmapsto[\mathbf{R}_\exists]{} B \sqsubseteq \exists S.C \xmapsto[\mathbf{R}_\sqsubseteq]{} B \sqsubseteq C$$

Methods do not have to be tied to a particular inference system

Problem statement

$$\frac{A \sqsubseteq \exists R.B, \quad B \sqsubseteq D, \quad D \sqsubseteq \exists S.A, \quad \exists R.D \sqsubseteq C, \quad \exists S.C \sqsubseteq C, \quad C \sqsubseteq \exists R.A}{}$$

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$C \sqsubseteq C$ $\quad$ $C \sqsubseteq \exists R.A$

How to re-construct proofs of $B \sqsubseteq C$ with minimal effort?
(e.g. without full tracing)

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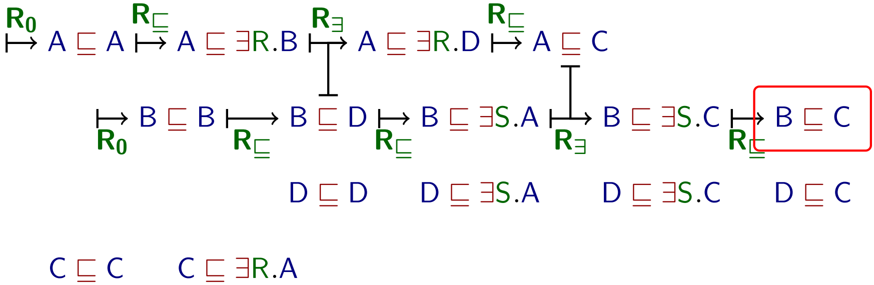

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Goal-directed tracing: main ideas

$A \sqsubseteq \exists R.B,\ B \sqsubseteq D,\ D \sqsubseteq \exists S.A,\ \exists R.D \sqsubseteq C,\ \exists S.C \sqsubseteq C,\ C \sqsubseteq \exists R.A$

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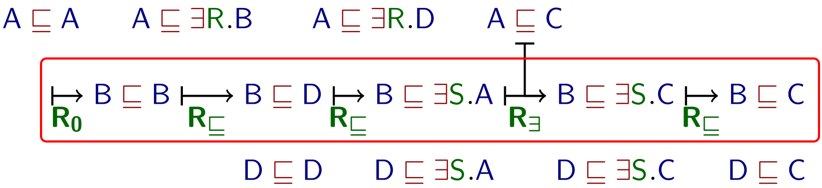

To find inferences for $B \sqsubseteq C$ it is sufficient to re-apply (and record) inferences applicable to subsumptions of the form $B \sqsubseteq X$

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Other subsumptions can be used, e.g., $A \sqsubseteq C$, but not derived

Goal-directed tracing: example

$A \sqsubseteq \exists R.B$, $B \sqsubseteq D$, $D \sqsubseteq \exists S.A$, $\exists R.D \sqsubseteq C$, $\exists S.C \sqsubseteq C$, $C \sqsubseteq \exists R.A$

Goal: reconstruct proofs of $B \sqsubseteq C$

Given: computed subsumptions (without tracing information)

$C \sqsubseteq C$ $\quad$ $C \sqsubseteq \exists R.A$

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Then, if needed, we can recursively request inferences for $A \sqsubseteq C$

Goal-directed tracing: example

$A \sqsubseteq \exists R.B$, $B \sqsubseteq D$, $D \sqsubseteq \exists S.A$, $\exists R.D \sqsubseteq C$, $\exists S.C \sqsubseteq C$, $C \sqsubseteq \exists R.A$

Goal: reconstruct proofs of $B \sqsubseteq C$

Given: computed subsumptions (without tracing information)

$C \sqsubseteq C$ $\quad C \sqsubseteq \exists R.A$

$\overset{\mathbf{R_0}}{\longmapsto} A \sqsubseteq A$ $\quad A \sqsubseteq \exists R.B$ $\quad A \sqsubseteq \exists R.D$ $\quad A \sqsubseteq C$

$\underset{\mathbf{R_0}}{\longmapsto} B \sqsubseteq B \underset{\mathbf{R_\sqsubseteq}}{\longmapsto} B \sqsubseteq D \underset{\mathbf{R_\sqsubseteq}}{\longmapsto} B \sqsubseteq \exists S.A \underset{\mathbf{R_\exists}}{\longmapsto} B \sqsubseteq \exists S.C \underset{\mathbf{R_\sqsubseteq}}{\longmapsto} B \sqsubseteq C$

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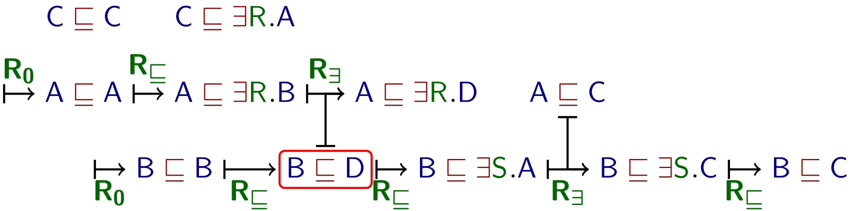

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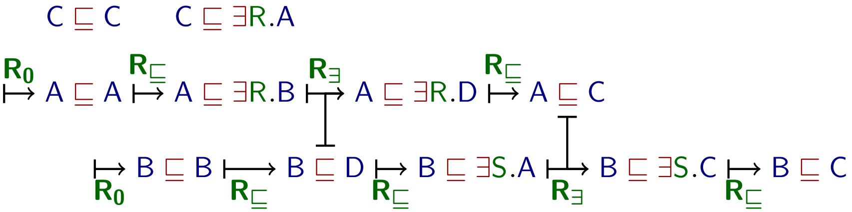
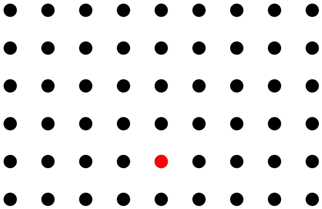


Then, if needed, we can recursively request inferences for $A \sqsubseteq C$

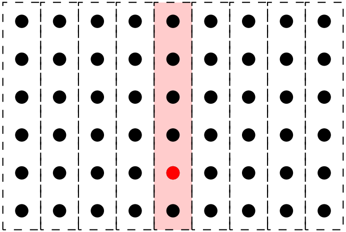
If we request inferences for $C \sqsubseteq \exists R.A$, we won't have to trace subsumptions with A or B on the left

Goal-directed tracing: the general procedure



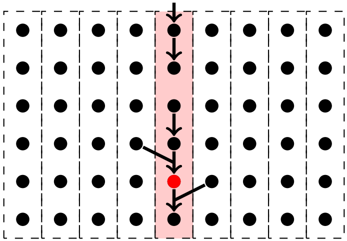
Goal-directed tracing: the general procedure

- partition the set of all derived expressions



Goal-directed tracing: the general procedure

- partition the set of all derived expressions
- apply and trace all inferences within the target partition



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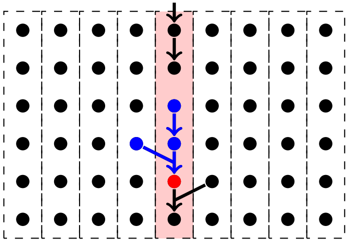

Goal-directed tracing: the general procedure

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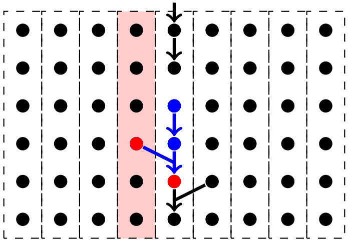
- identify expressions used as a part of a proof
- recursively trace such expressions as new targets



$\mathcal{EL}$ partitions for goal-directed tracing

We use the left hand-side as the partition ID for each subsumption

$$C \sqsubseteq D \mapsto C$$

$$\mathbf{R_0}\ \frac{}{C \sqsubseteq C} \qquad \mathbf{R}_\sqcap^-\ \frac{C \sqsubseteq D_1 \sqcap D_2}{C \sqsubseteq D_1 \quad C \sqsubseteq D_2} \qquad \mathbf{R}_\sqsubseteq\ \frac{C \sqsubseteq D}{C \sqsubseteq E} : D \sqsubseteq E \in \mathcal{O}$$

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$$C \sqsubseteq D \mapsto C$$

The key property of this partitioning function:

- each inference's conclusion belongs to the same partition as one of the premises.
- won't miss anything when applying rules only to premises in the target partition (using other expressions as set of support).

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Common $\mathcal{EL}^+$ ontologies: SNOMED CT, EL-GALEN, GO-EXT

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Hypothesis for goal-directed tracing evaluation:

- few partitions are traced
- few inferences are re-applied

Goal-directed tracing: experimental setup

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Goal-directed tracing: experimental setup

- classify the ontology without tracing
- separately trace each subsumption between named concepts (and average over all of them)
- with full recursive proof unwinding
- separately measure traced and used inferences
  - used means “used as a part of a proof”

Goal-directed tracing: results

GO: 5.7M inferences GALEN: 3.9M SNOMED CT: 54.5M

Ontology	Total partitions	# of traced partitions	# of traced inferences	# of inferences used in proofs	# of used $\sqsubseteq$ axioms	Time (in ms.)
GO	46,8K	3.7	456.2	94.9	18.6	2.0
GALEN	25.9K	1.9	414.4	21.7	5.2	1.9
SNOMED CT	371.6K	8.6	788.3	385.9	9.7	10.1

Goal-directed tracing is fast.

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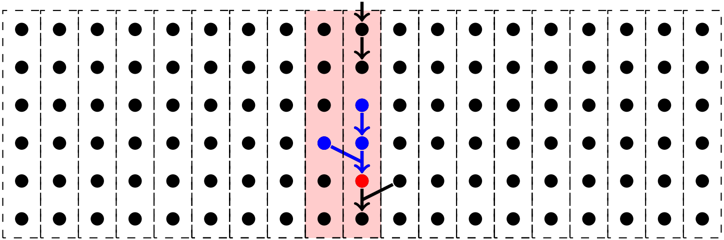
Only few partitions/inferences are traced.



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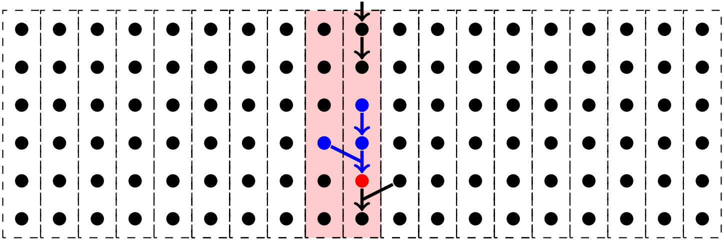
More inferences are stored than used in proofs. Doesn't matter.



Summary

Presented a new method for tracing which is

- agnostic to the rule system used
  (efficiency depends on partitioning)
- very simple and efficient for $\mathcal{EL}$
- does not slow down the base procedure

Exploits the granularity property of the $\mathcal{EL}$ procedure, as for:

- concurrency
- incremental reasoning

Method can be used for:

- proof reconstruction, understanding, verification...
- interactive step-wise debugging
- speeding up justifications

Questions?

Tracing and justifications are different

Compute different things

Justifications:

- minimal subsets entailing the result
- intractable

Tracing:

- compact representation of all proofs
- not necessarily only minimal proofs!
- tractable

Tracing and justifications can be used together

Approach 1: use traces to reduce search space

- to the union over all axioms used in proofs
- i.e. like modules, just smaller

Approach 2: directly read justifications off proofs

- not every minimal proof corresponds to a justification
- ...but every justification corresponds to a proof!
- enumerate (via unwinding) all proofs and read justifications
- can be done offline, without the reasoner

Tracing and locality-based modules

Ontology modularity often used with justifications

- any justification is a subset of the module for the subsumption
- modules can be efficiently computed (and are usually small)
- justifications can be found inside the module

That's similar to searching for justifications inside the traces

Tracing and locality-based modules

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That's similar to searching for justifications inside the traces

And there's a relationship:

- modules cannot be smaller than traces
  if $A \sqsubseteq B$ is derived then $M_B(\mathcal{O}) \subseteq M_A(\mathcal{O})$
- but they can be significantly larger

Tracing and locality-based modules

The GALEN case: the entire anatomical part is connected

$A_0 \equiv B_0 \sqcap \exists R.A_1, \ldots, A_i \equiv B_i \sqcap \exists R.A_{i+1}$

The module for A_0 will contain modules for B_i and A_i

More than 10K axioms on average for EL-GALEN

The trace for A_0 will not if partitions for most of A_i and B_i do not have inferences used in any proof