Kernel Learning for SVM-based System Identification.

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1. Outline

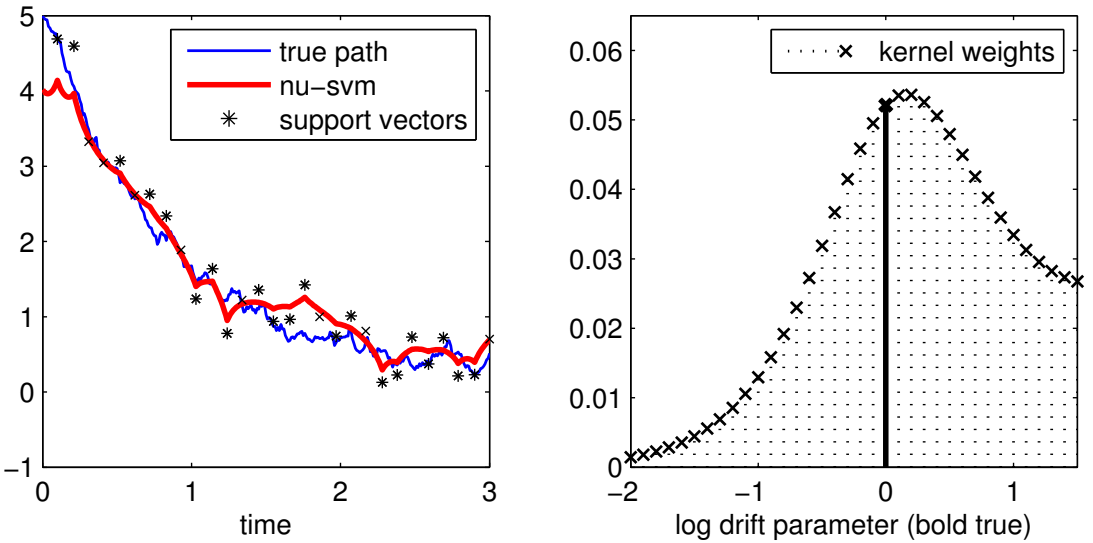
- Application of multiple kernel learning to the problem of system identification for temporal data.
- Aim: To learn the covariance function of a stochastic process.
- Computationally simple approach using out of the box machine learning methods.
- Proposed method for multidimensional, non-linear, non-stationary systems.

2. Simple ν-SVM Based Covariance Learning

- *Primitive* Ornstein-Uhlenbeck (OU) problem,

$$dX = -\gamma X dt + \sigma dW; \quad K(s,t) \propto \sigma^2 \exp(-\gamma|s-t|).$$

- Simple MKL ν-SVM applied to discrete time (noisy) observations of OU path



- Problem: Most dynamical systems are multidimensional, non-linear and/or non-stationary.

3. Vector-valued ν-SVM

- Kernel $\mathbf{K} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^d \times \mathbb{R}^d$ and ϵ-insensitive loss $L_{\epsilon,p}(\mathbf{y}, \hat{\mathbf{y}}) = \max(0, ||\mathbf{y} - \hat{\mathbf{y}}||_p - \epsilon)$.
- Vector valued ν-SVM dual form ($\bar{\gamma}_i = \gamma_i^+ - \gamma_i^-$)

$$\max_{\{\gamma_i^+, \gamma_i^-\}} \quad \sum_{i=1}^n \bar{\gamma}_i^\top \mathbf{y}_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \bar{\gamma}_i^\top \mathbf{K}(t_i, t_j) \bar{\gamma}_i$$

$$\text{subject to} \quad \sum_{i=1}^n \bar{\gamma}_i = 0; \quad \sum_{i=1}^n ||\gamma_i^{(\pm)}||_{\frac{p}{p-1}} \leq \nu; \quad ||\gamma_i^{(\pm)}||_{\frac{p}{p-1}} \leq \frac{C}{n}.$$

- Simple MKL still valid for $\mathbf{K}_{\boldsymbol{\eta}} = \sum_{j=1}^p \eta_j \mathbf{K}_j$,

$$\frac{\partial \text{DUAL}(\boldsymbol{\eta}, \{\gamma_i^+, \gamma_i^-\})}{\partial \eta_k} = -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \bar{\gamma}_i^\top \mathbf{K}_k(t_i, t_j) \bar{\gamma}_i.$$

4. GP Approximation of Non-Linear Systems

- Aim: Local GP approximation of a non-linear path.
- Non-stationary kernel, built from RBFs $\psi_i(t_i - t)$ at observation times $\{t_i\}$,

$$\mathbf{K}_{\boldsymbol{\Theta}}(s,t) = \sum_{j=1}^{p} \varphi_j(t|\boldsymbol{\Theta})\varphi_j(s|\boldsymbol{\Theta})\mathbf{K}_j(s,t),$$

$$\varphi_j(t|\boldsymbol{\Theta}) = \sum_{i=1}^{n} \Theta_{i,j}\psi_i(t_i - t); \quad \Theta_{i,j} \in \mathbb{R}_+; \quad \sum_{j=1}^{p} \Theta_{i,j} = 1.$$

- Equivalent to *Localised* MKL; optimisation non-trivial (work in progress), output $\widetilde{\boldsymbol{\Theta}}$, $\{\bar{\gamma}_i^*\}$, ϵ^*,

$$\boldsymbol{\mu}(\cdot) = \sum_{i=1}^{n} \bar{\gamma}_i^* \mathbf{K}_{\widetilde{\boldsymbol{\Theta}}}(t_i, \cdot).$$

5. Statistical Analysis

- GP $\mathbf{X}(\cdot) \sim GP(\boldsymbol{\mu}(\cdot), \mathbf{K}_{\widetilde{\boldsymbol{\Theta}}}(\cdot,\cdot))$ governed by *linear* SDE

$$d\mathbf{X}(t) = \widetilde{\mathbf{f}}_L(\mathbf{X}(t), t)dt + \Sigma d\mathbf{W}(t).$$

- *Any* Q governed by $d\mathbf{Z}(t) = \mathbf{g}(\mathbf{Z}(t), t)dt + \sqrt{\Sigma}d\mathbf{W}(t)$,

$$\mathrm{KL}(GP||Q) = \frac{1}{2}\int_0^T \mathop{\mathbb{E}}_{\mathbf{X}\sim GP(\boldsymbol{\mu},\mathbf{K}_{\widetilde{\boldsymbol{\Theta}}})} \left[||\widetilde{\mathbf{f}}_L(\mathbf{X}(t), t) - \mathbf{g}(\mathbf{X}(t), t)||_\Sigma^2\right] dt.$$

- Assume $\{(\mathbf{y}_i, t_i)\}$ iid; define

$$e_{GP}(\mathbf{y}, t) = \mathop{\mathbb{E}}_{\mathbf{X}\sim GP(\boldsymbol{\mu},\mathbf{K}_{\widetilde{\boldsymbol{\Theta}}})} [\mathbb{1}(L_{\epsilon^*,p}(\mathbf{y}, \mathbf{X}(t)) > 0)];$$

$$e_{GP} = \mathbb{E}_{(\mathbf{y},t)\sim D}[e_{GP}(\mathbf{y}, t)]; \quad \widehat{e}_{GP} = \frac{1}{n}\sum_{i=1}^{n}[e_{GP}(\mathbf{y}_i, t_i)].$$

With probability at least $1-\delta$ over the draw of the sample,

$$\mathrm{KL}_{Bernoulli}(\widehat{e}_{GP}||e_{GP}) \leq \frac{\mathrm{KL}(GP(\boldsymbol{\mu}, \mathbf{K}_{\widetilde{\boldsymbol{\Theta}}})||Q) + \ln\frac{n+1}{\delta}}{n}.$$

References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