Operator Induced Multi-Task Gaussian Processes for Solving Differential Equations

Arman Melkumyan
Australian Centre for Field Robotics, University of Sydney, Australia

Task: Solve ordinary and partial differential equations with initial and/or boundary conditions

Equation: $u'(x) + u(x)\sin x = \sin^3 x$ in the region $x \in (0.5\pi, 2.5\pi)$. Boundary condition: $u(\pi/2) = 0$

Equation: $u_{xx} + u_{yy} = 1$ in the region $x^2 + y^2 < 1$. Boundary condition: $u(x,y) = 0$ on the circle $x^2 + y^2 = 1$

Equation: $u_{xx} - 0.25u_{tt} = 0$ in the region $x \in (0,\pi)$, $t \in (0,\pi)$. BC: $u(0,t) = u(\pi,t) = 0$ on the interval $t \in (0,\pi)$

Initial conditions: $u(x,0) = 0.1\sin^3 x$, $u_t(x,0) = 0$ on the interval $x \in (0,\pi)$

General formulation:

Equation: $L_{1,\mathbf{x}}[u(\mathbf{x})] = h_1(\mathbf{x})$ in the region Ω_1

Boundary and/or initial conditions: $L_{l,\mathbf{x}}[u(\mathbf{x})] = h_l(\mathbf{x})$ in the regions $\Omega_2, \ldots, \Omega_M$

Proposed approach:

- Assume the solution to be a GP with a particular covariance function
- Each linear operator will result into a new GP with a different covariance function
- Consider each equation as a single GP within a multi-task GP framework
- Discretize each equation to create a dataset for all the GP tasks
- Apply multi-task GP learning to optimize the hyper-parameters
- Conduct GP inference to obtain the numerical solution

Test 1. First Order ODE with Variable Coefficients

$$u'(x)+u(x)\sin x=\sin^3 x \quad \forall x\in(0.5\pi,2.5\pi). \quad \text{BC: } u(\pi/2)=0$$

Test 2. Second Order ODE with Constant Coefficients

$$u''(x)+\omega^2 u(x)=0 \quad \forall x\in(0,2\pi). \quad \text{BC } u(0)=0,\ u'(0)=1$$

Test 3. Second Order ODE with Variable Coefficients

$$u''+r^{-1}u'-r^{-2}u=-ar \quad \forall r\in(0,R). \quad \text{BC: } u(0)=0,\ \sigma_{rr}(R)=0$$

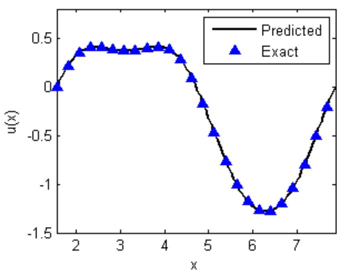
Test 4. Second Order Hyperbolic Partial Differential Equation

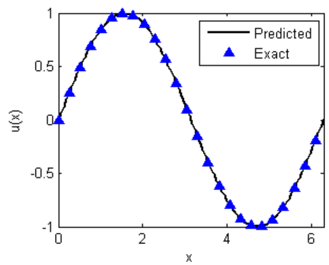
$$u_{xx}-0.25u_{tt}=0 \quad \forall x\in(0,\pi),\ t\in(0,\pi). \quad \text{BC: } u(0,t)=u(\pi,t)=0 \quad \forall t\in(0,\pi)$$

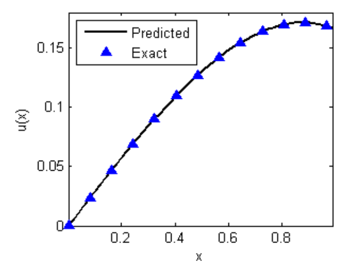
$$\text{IC}: u(x,0)=0.1\sin^3 x,\ u_t(x,0)=0 \quad \forall x\in(0,\pi)$$

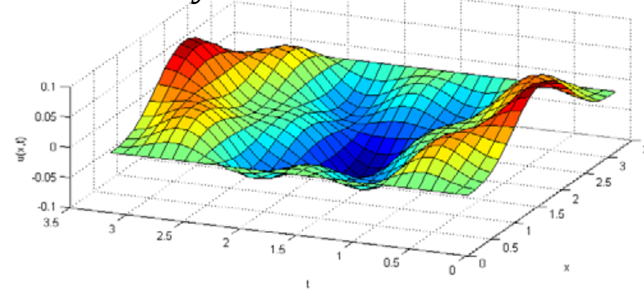
Test 5. Second Order Elliptic Partial Differential Equation

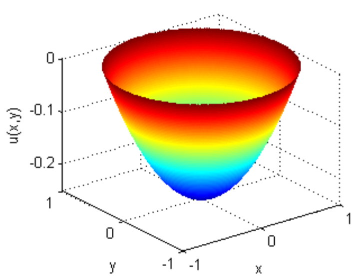
$$u_{xx}+u_{yy}=1 \quad \forall x^2+y^2<1. \quad \text{BC: } u(x,y)=0 \text{ when } x^2+y^2=1$$











	Test 1	Test 2	Test 3	Test 4	Test 5
Equation type	1st order ODE, variable coefficients	2nd order ODE, constant coefficients	2nd order ODE, variable coefficients	2nd order PDE, hyperbolic	2nd order PDE, elliptic
# of discret. points	11	10	9	265	13
MAE	3.847E-6	4.120E-10	9.663E-11	1.587E-6	1.525E-11