Game Dynamics with Learning and Evolution of Universal Grammar

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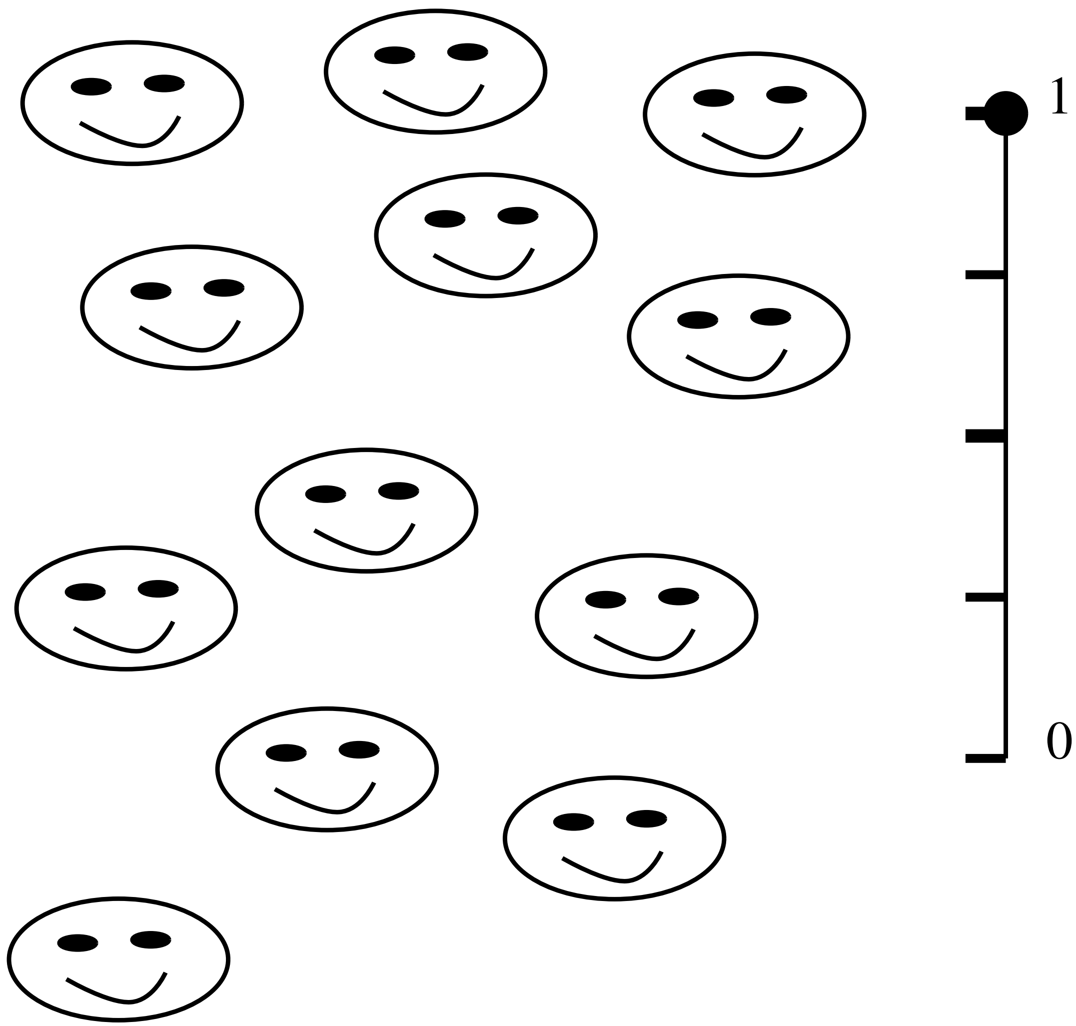
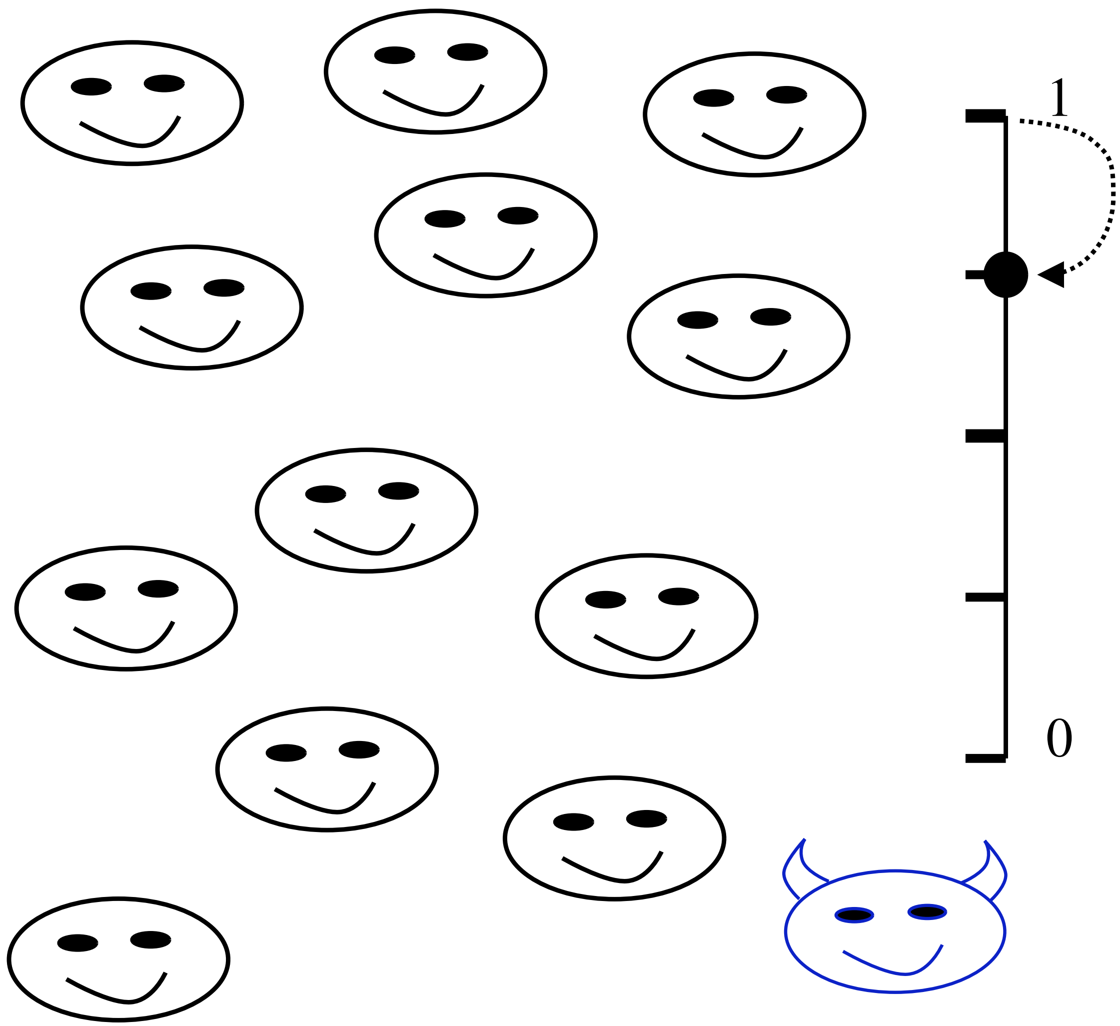
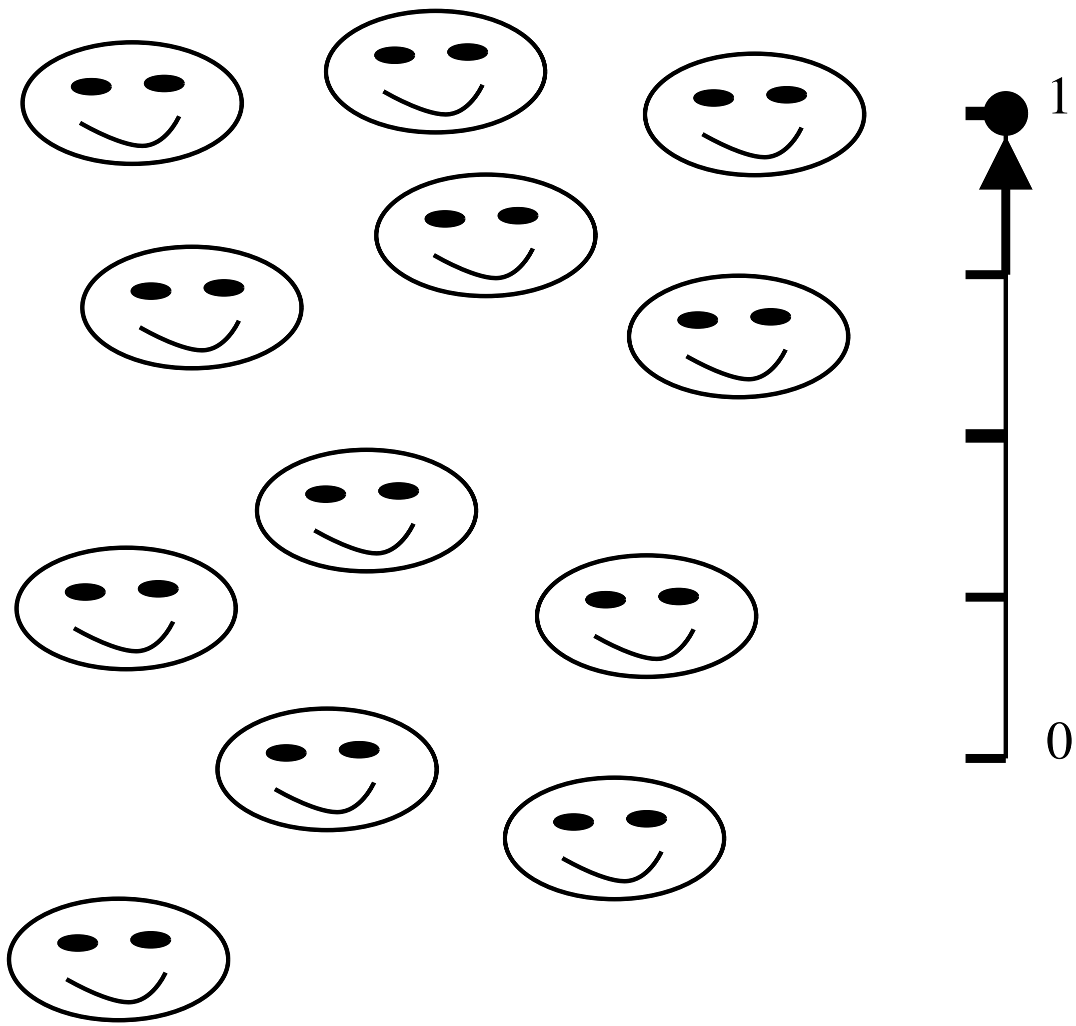
Start: genetically homogeneous



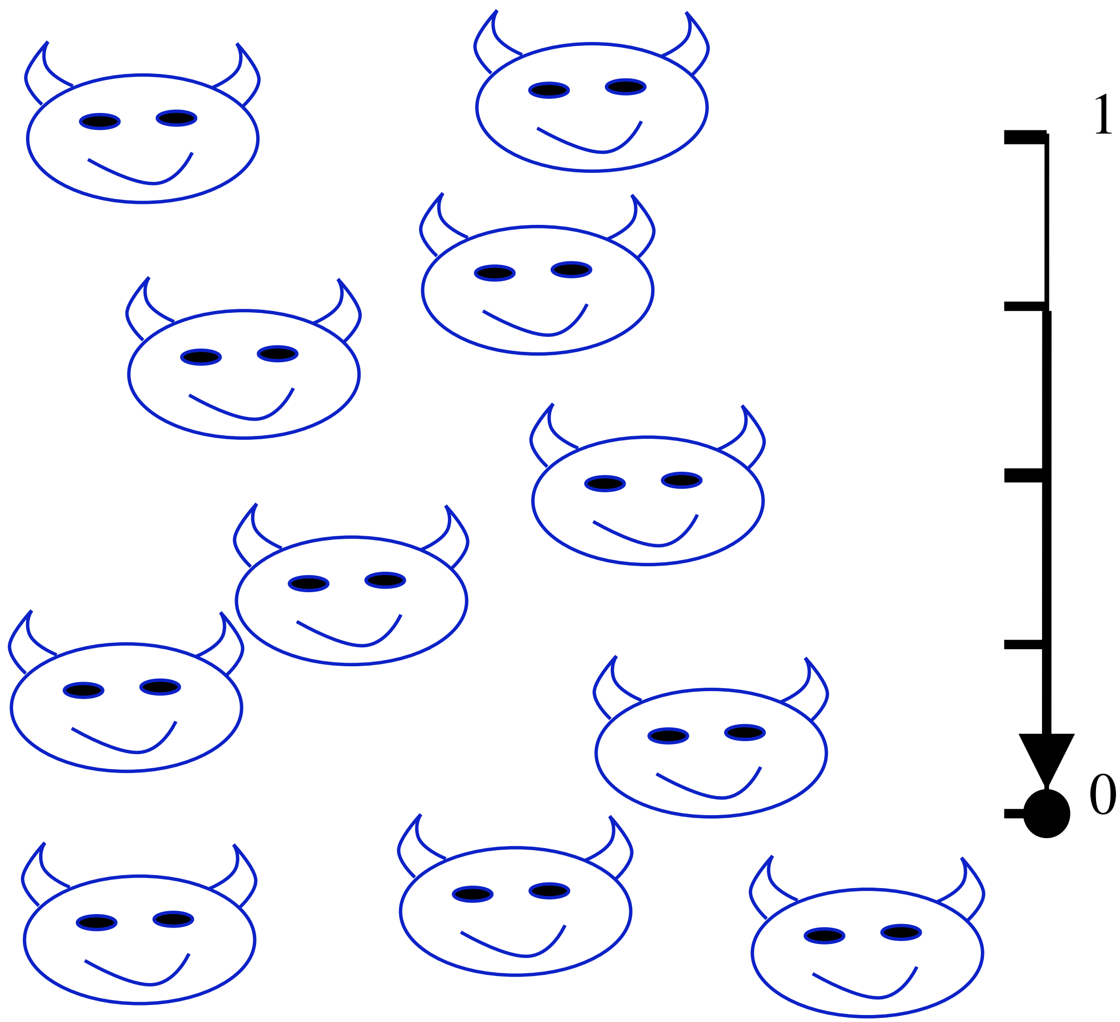
Invasion: different gene



Invasion dies out



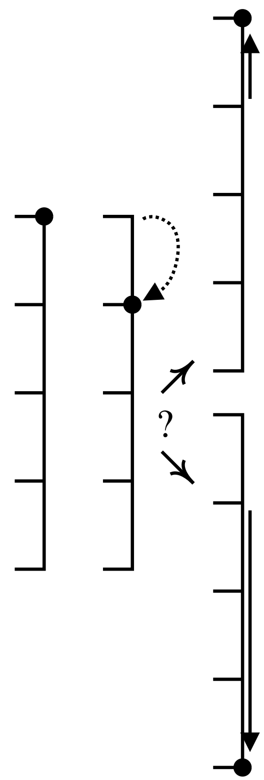
Invasion succeeds



Genetic changes

- UG = set of allowed grammars + learning process
- Can U_2 invade U_1? Depends on…
  - …grammars allowed
  - …payoff matrix
  - …learning processes
  - …population state

$\Rightarrow$ Can depend on *when*



Traditional game dynamics

- Nash equilibrium: Best reply to itself
- Evolutionarily Stable Strategy (ESS): Cannot be invaded
- Not sufficient for *metastrategy* game: Contest between strategies for selecting strategies for an abstract game
- Payoff for U_2 playing U_1 is not a constant

Language model

- $U_1, \ldots, U_N$ = universal grammars
- $G_1, \ldots, G_n$ = all possible grammars
- $B = (b_{i,j})$ = payoff for G_i when talking to G_j
- $Q_{i,j,K} = \mathbb{P}\left\{ \begin{array}{l} \text{a child of a speaker of } G_i \text{ ends} \\ \text{up speaking } G_j\text{, given that both} \\ \text{have } U_K \end{array} \right\}$

Population structure

- $x_{j,K}$ = fraction of the population with U_K that speaks G_j
- y_K = fraction with U_K

$$y_K = \sum_j x_{j,K}$$

- w_j = fraction that speaks G_j

$$w_j = \sum_K x_{j,K}$$

Dynamics

- F_j = weighted payoff to G_j

$$F = Bw \quad \Longleftrightarrow \quad F_j = \sum_k b_{j,k} w_k$$

- ϕ = average payoff

$$\phi = w^T F = \sum_j F_j w_j$$

$$\begin{aligned} \dot{x}_{j,K} &= \sum_i F_i x_{i,K} Q_{i,j,K} - \phi x_{j,K} \\ &= x_{j,K}(F_j Q_{j,j,K} - \phi) + \sum_{i \neq j} F_i x_{i,K} Q_{i,j,K} \end{aligned}$$

Extended dynamics

- Competition among UGs:

$$\dot{y}_K = \sum_j \dot{x}_{j,K}$$

$$= \left(\sum_i F_i x_{i,K} \left(\sum_j Q_{i,j,K} \right) \right) - \phi \left(\sum_j x_{j,K} \right)$$

$$= \sum_i F_i x_{i,K} - \phi y_K$$

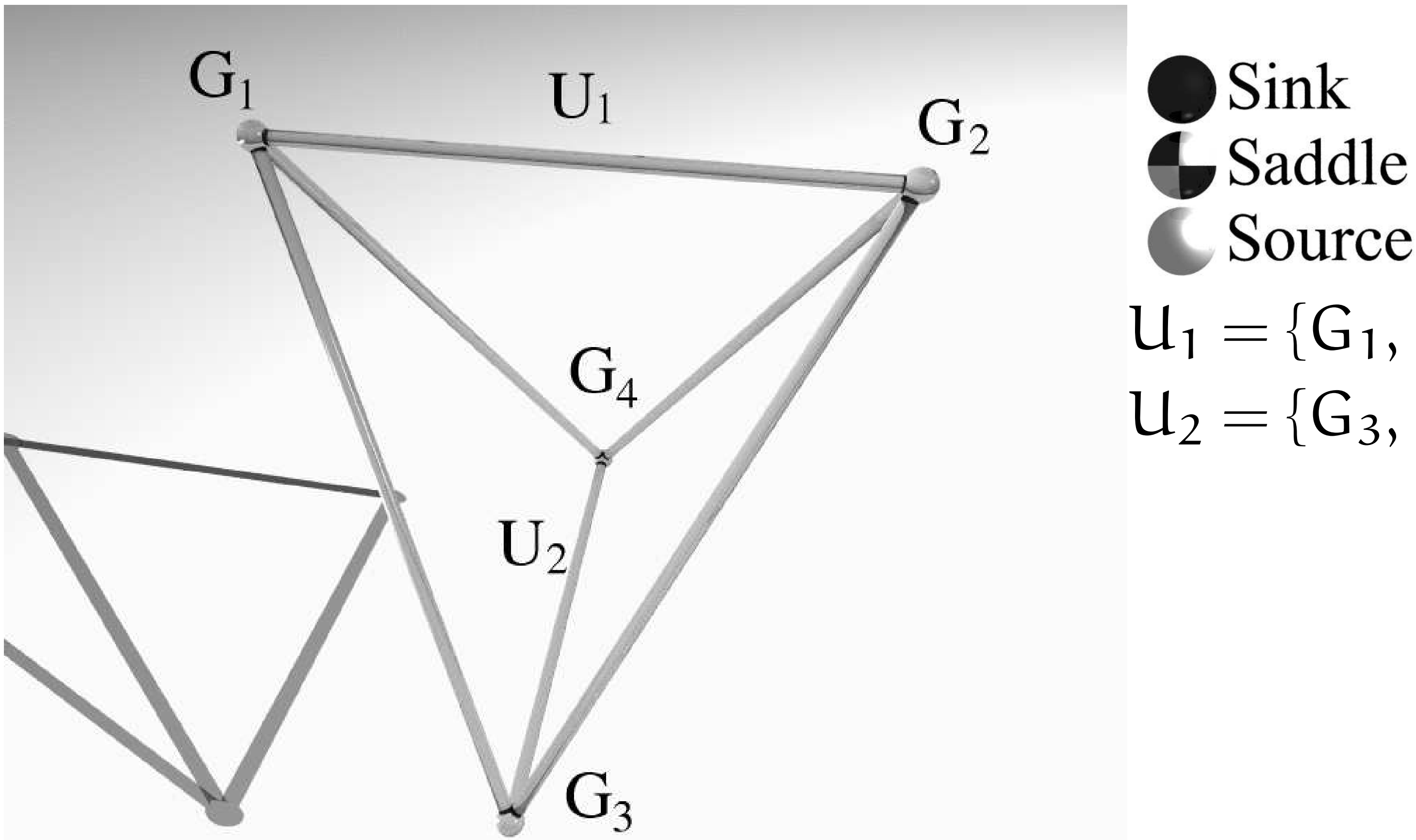
A tractable case

- $U_1 = \{G_1, G_2\}$, $U_2 = \{G_3, G_4\}$

$\Rightarrow$ $x_{3,1} = x_{4,1} = x_{1,2} = x_{2,2} =$ all fixed at 0

- Leaves: $x_{1,1}$ $x_{2,1}$ $x_{3,2}$ $x_{4,2}$
- General payoff matrix $B = (b_{ij})$
- General learning matrix Q

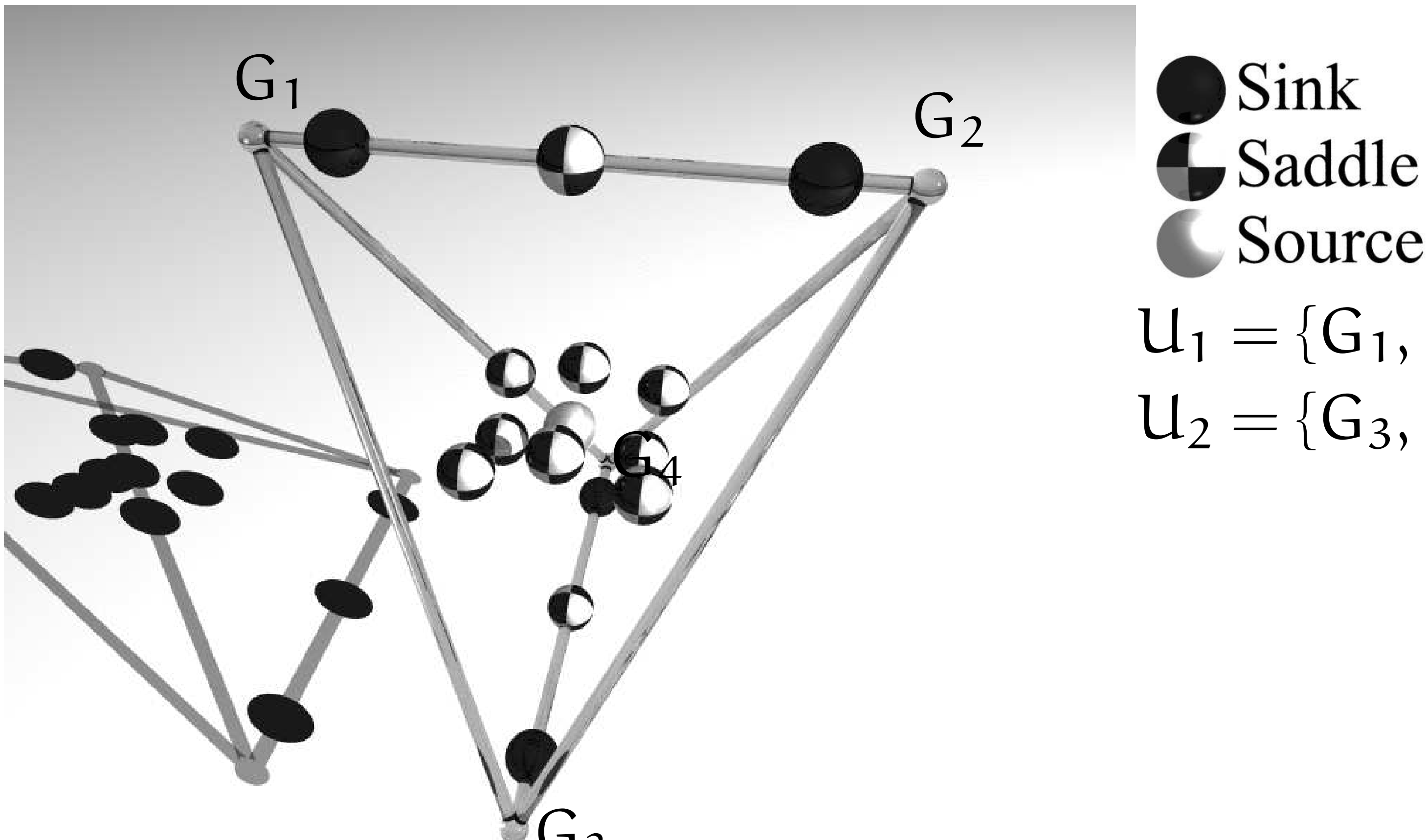
Notation convention



$U_1 = \{G_1, G_2\}$

$U_2 = \{G_3, G_4\}$

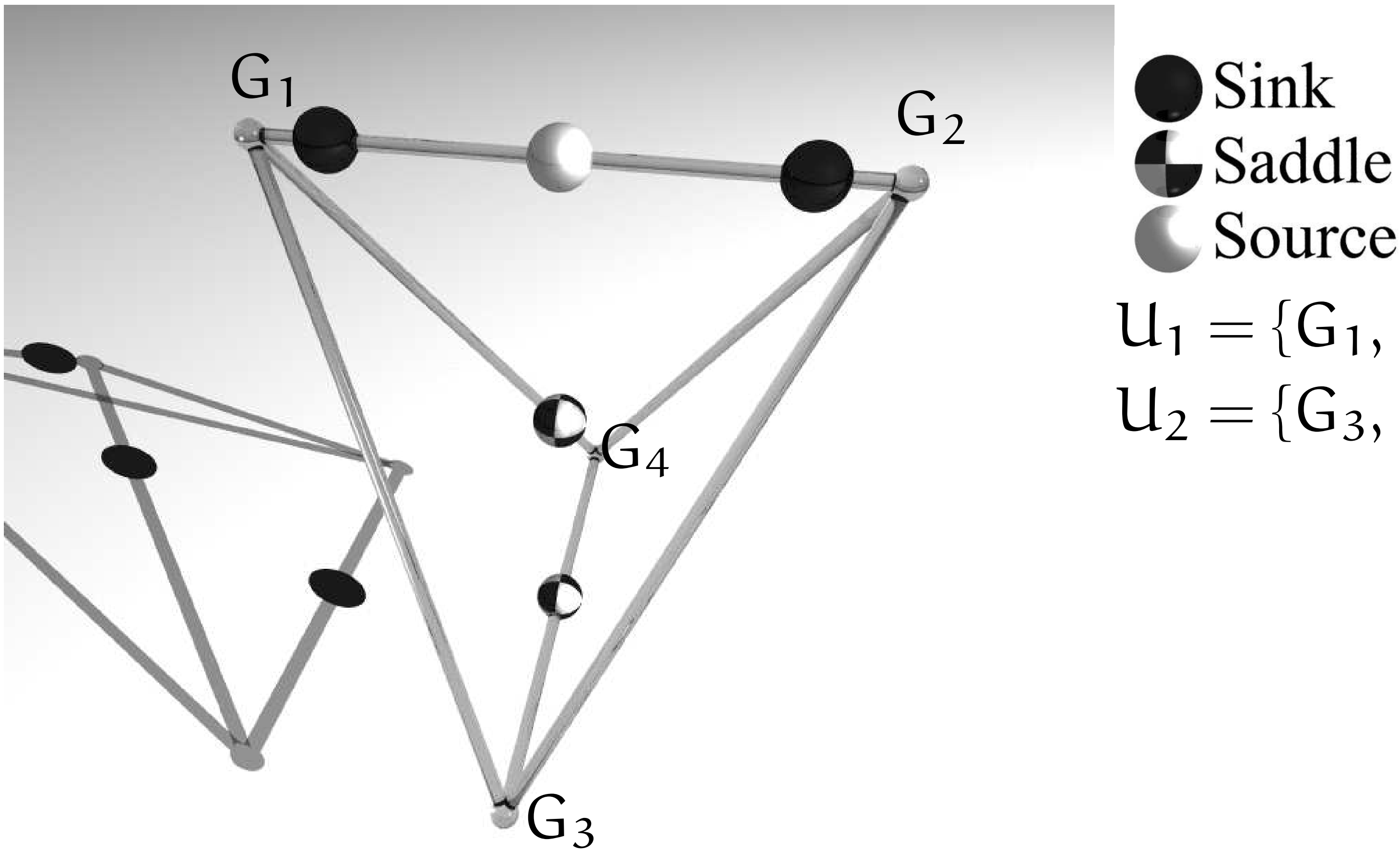
Exclusion



$U_1 = \{G_1, G_2\}$

$U_2 = \{G_3, G_4\}$

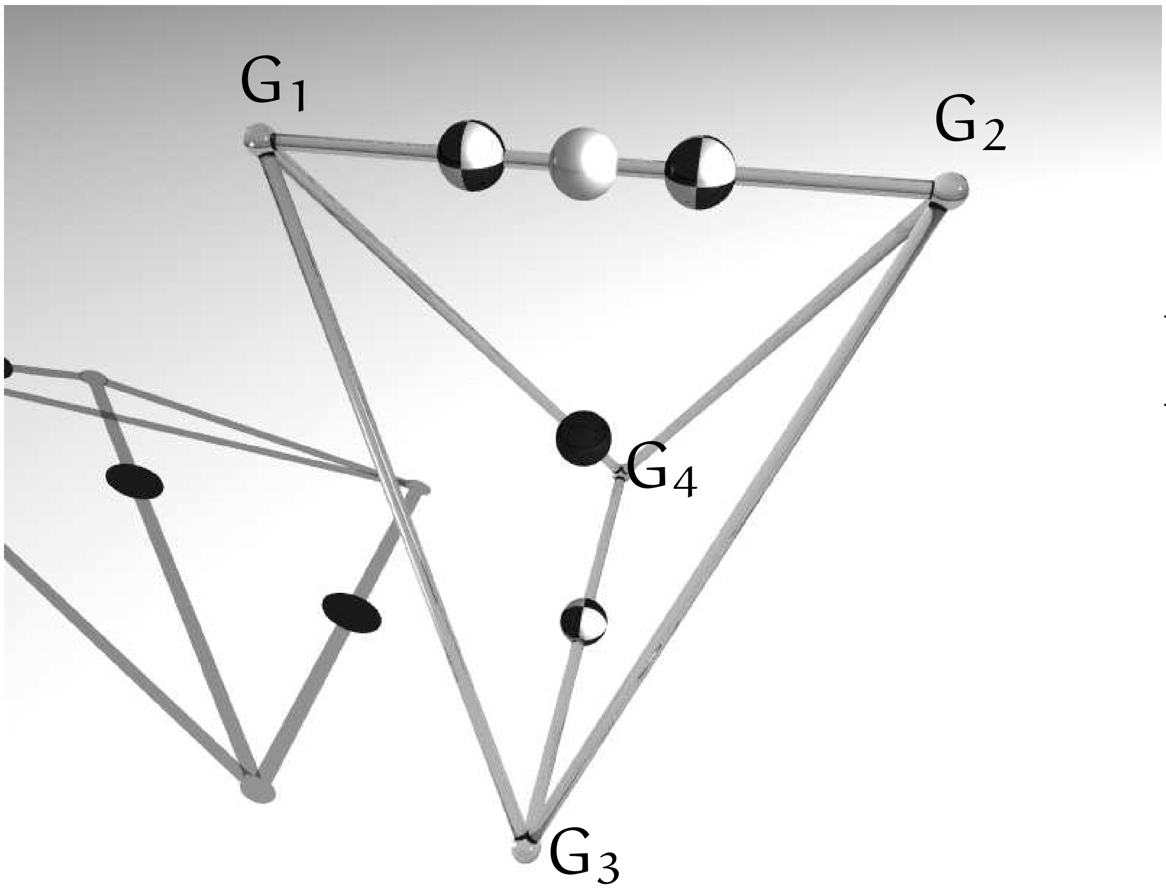
Dominance



$U_1 = \{G_1, G_2\}$

$U_2 = \{G_3, G_4\}$

Symmetric coexistence

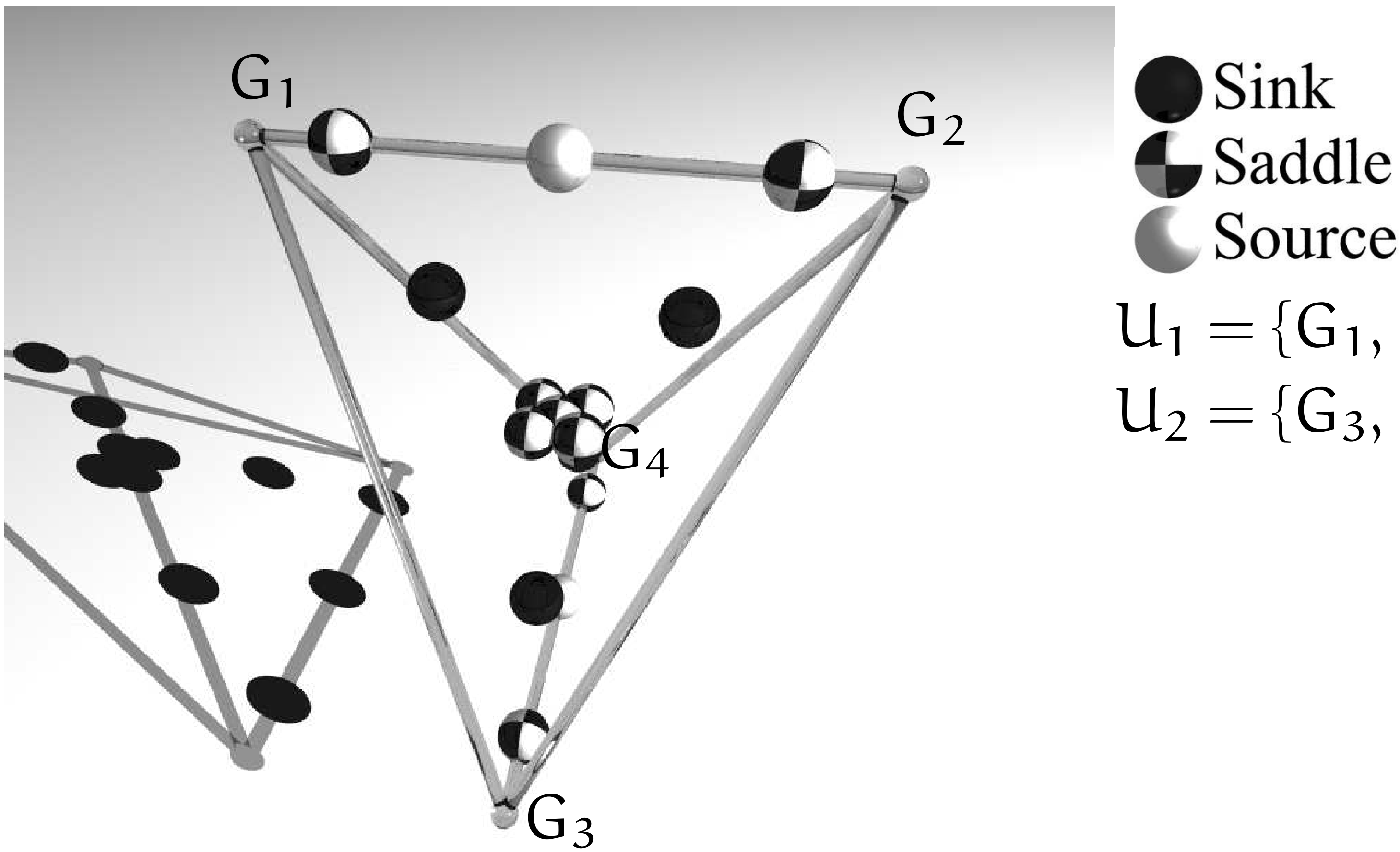


Sink
Saddle
Source

$U_1 = \{G_1, G_2\}$

$U_2 = \{G_3, G_4\}$

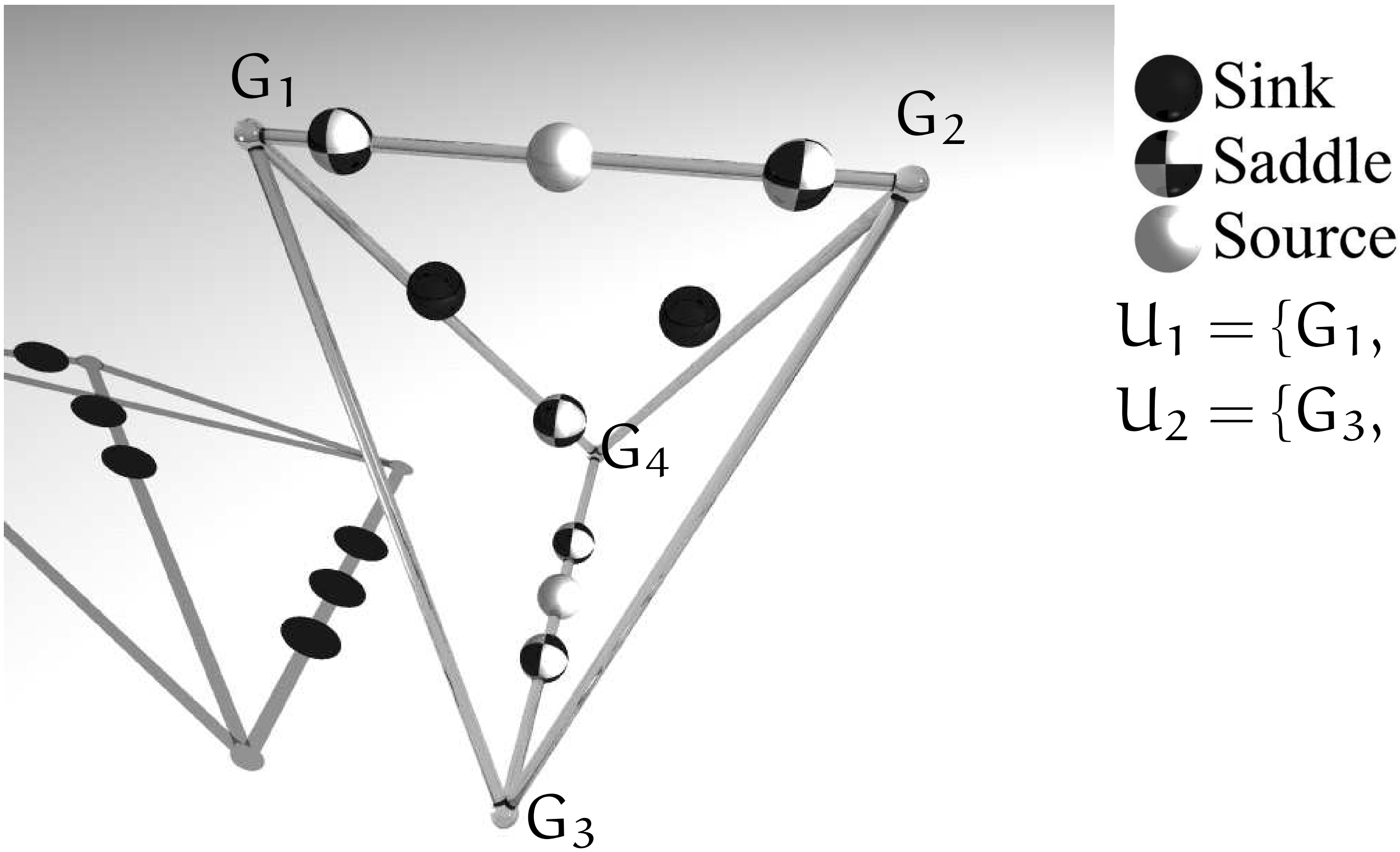
Asymmetric coexistence



$U_1 = \{G_1, G_2\}$

$U_2 = \{G_3, G_4\}$

Asymmetric coexistence



$U_1 = \{G_1, G_2\}$

$U_2 = \{G_3, G_4\}$

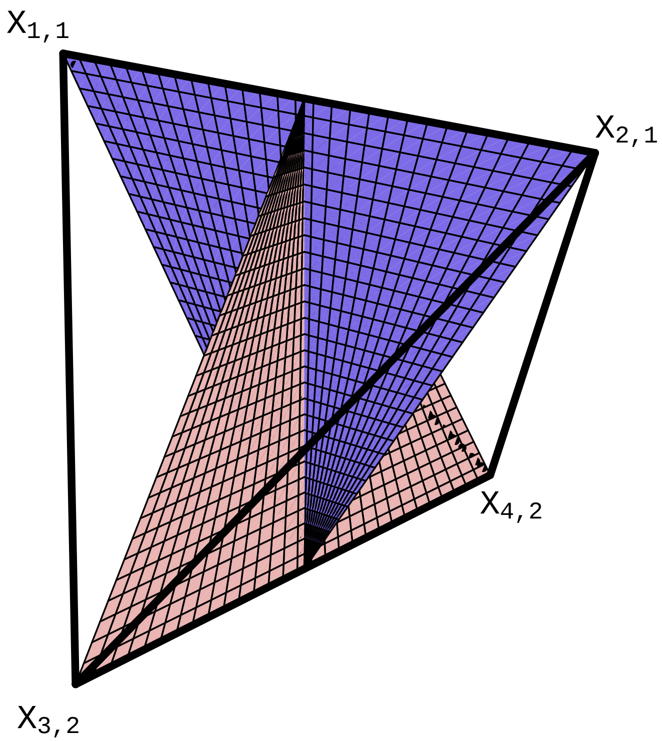
Goal

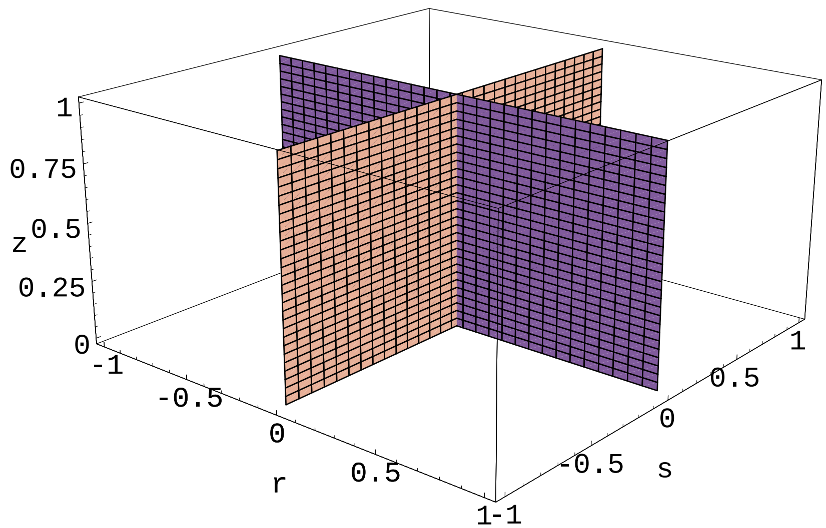
- Goal: Find conditions on B that imply that U_2 cannot invade a population of all U_1, and U_1 cannot invade a population of all U_2

  $\Longleftrightarrow \{\text{all } U_1\}$ and $\{\text{all } U_2\}$ are attracting sets

- **Attracting set:**
  - closed
  - invariant under the dynamical system
  - neighborhood of points that tend to the set

Phase space





$$r = \frac{x_{2,1} - x_{1,1}}{x_{2,1} + x_{1,1}} \qquad -1 \leq r \leq 1$$

$$s = \frac{x_{4,2} - x_{3,2}}{x_{4,2} + x_{3,2}} \qquad -1 \leq s \leq 1$$

$$z = x_{1,1} + x_{2,1} = y_1 \qquad 0 \leq z \leq 1$$

New parameters

$$\alpha_0 = \frac{1}{2}(b_{11} + b_{12} + b_{21} + b_{22}) \qquad \alpha_1 = \frac{1}{2}(b_{11} - b_{12} - b_{21} + b_{22})$$

$$\alpha_2 = \frac{1}{2}(b_{11} + b_{12} - b_{21} - b_{22}) \qquad \alpha_3 = \frac{1}{2}(b_{11} - b_{12} + b_{21} - b_{22})$$

$$\beta_0 = \frac{1}{2}(b_{13} + b_{14} + b_{23} + b_{24}) \qquad \beta_1 = \frac{1}{2}(b_{13} - b_{14} - b_{23} + b_{24})$$

$$\beta_2 = \frac{1}{2}(b_{13} + b_{14} - b_{23} - b_{24}) \qquad \beta_3 = \frac{1}{2}(b_{13} - b_{14} + b_{23} - b_{24})$$

$$\gamma_0 = \frac{1}{2}(b_{31} + b_{32} + b_{41} + b_{42}) \qquad \gamma_1 = \frac{1}{2}(b_{31} - b_{32} - b_{41} + b_{42})$$

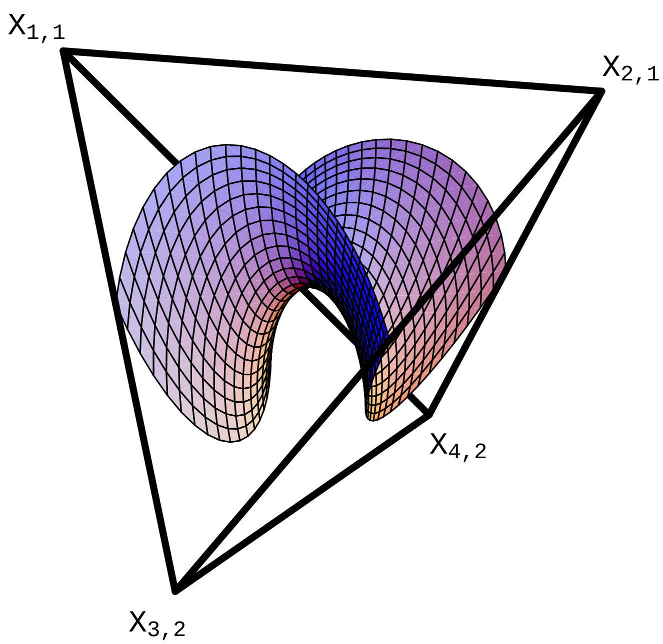
$$\gamma_2 = \frac{1}{2}(b_{31} + b_{32} - b_{41} - b_{42}) \qquad \gamma_3 = \frac{1}{2}(b_{31} - b_{32} + b_{41} - b_{42})$$

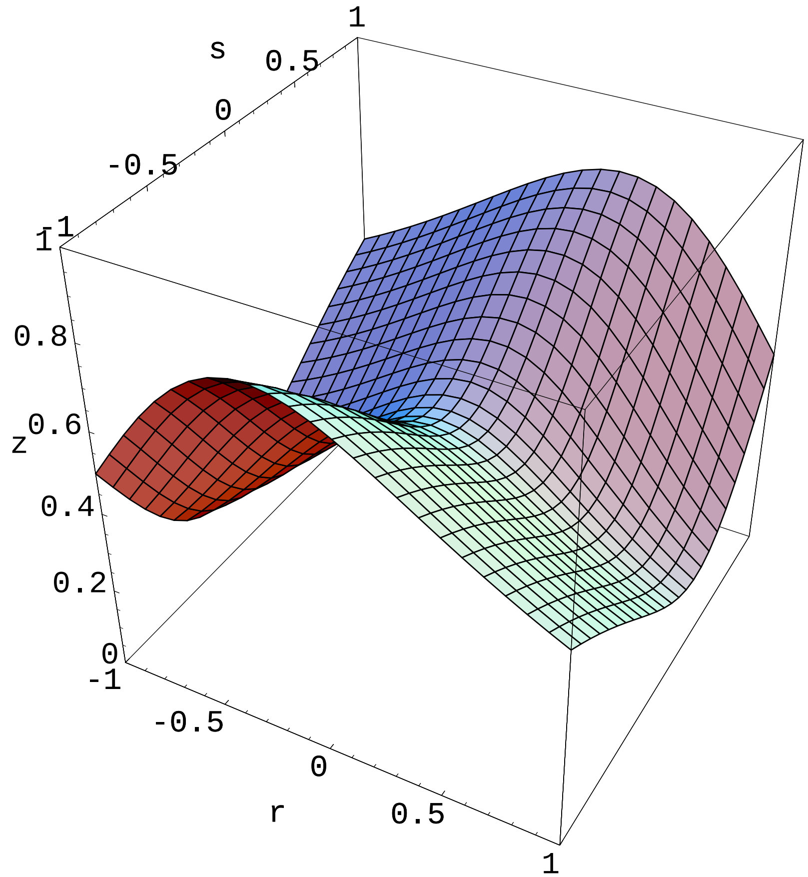
$$\delta_0 = \frac{1}{2}(b_{33} + b_{34} + b_{43} + b_{44}) \qquad \delta_1 = \frac{1}{2}(b_{33} - b_{34} - b_{43} + b_{44})$$

$$\delta_2 = \frac{1}{2}(b_{33} + b_{34} - b_{43} - b_{44}) \qquad \delta_3 = \frac{1}{2}(b_{33} - b_{34} + b_{43} - b_{44})$$

Where $\dot{z} = 0$: z null-clines

$$\dot{z} = \frac{1}{4}(z-1)zg(r,s,z)$$



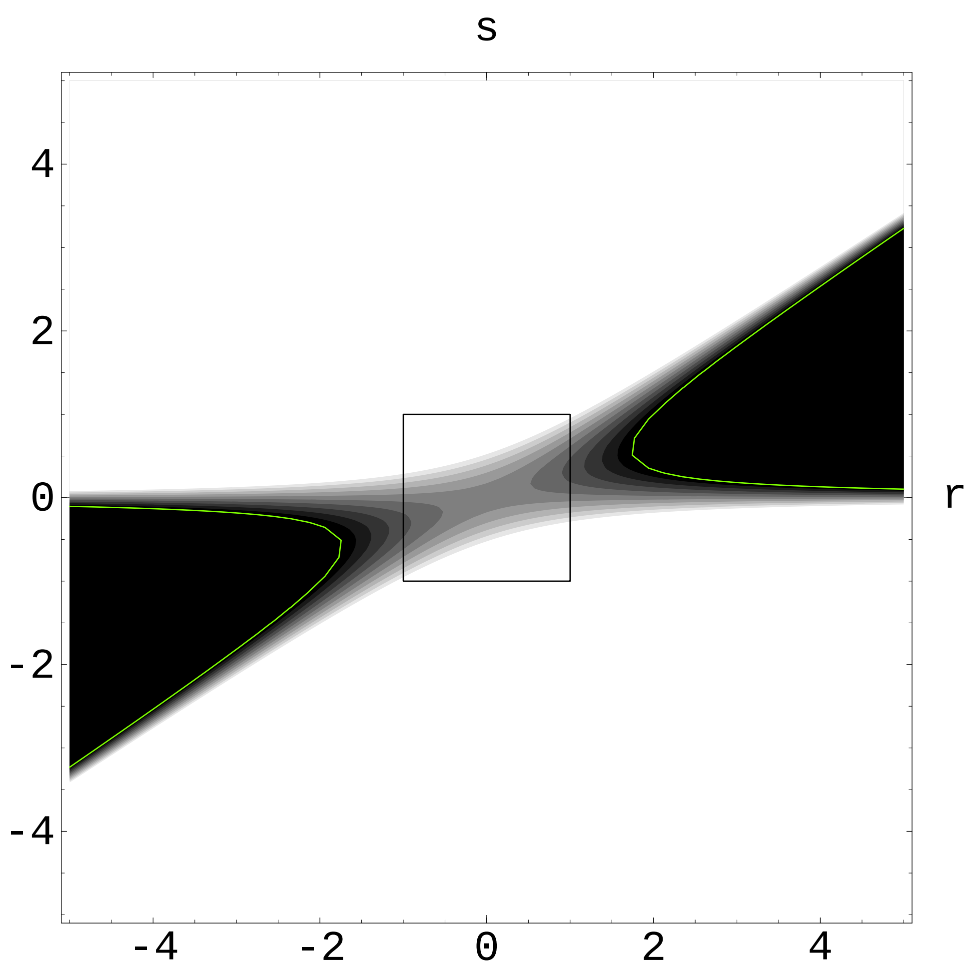



RALPH LAUREN
the nullcline collection

g

$$g(r,s,z) =$$
$$2\Big(-\beta_0 + \delta_0 + r\beta_2 + s(\beta_3 - \delta_2 - \delta_3)$$
$$- z(\alpha_0 - \beta_0 - \gamma_0 + \delta_0) - rs\beta_1$$
$$+ rz(\alpha_2 + \alpha_3 - \beta_2 - \gamma_3)$$
$$- sz(\beta_3 + \gamma_2 - \delta_2 - \delta_3)$$
$$+ s^2\delta_1 + rsz(\beta_1 + \gamma_1) - r^2 z\alpha_1 - s^2 z\delta_1\Big)$$

Non-intersection on bottom



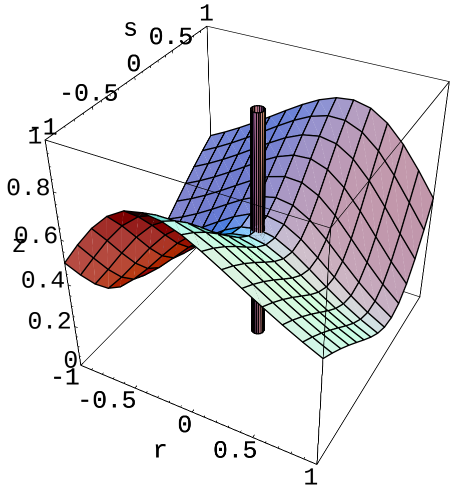
$g(r, s, 0)$

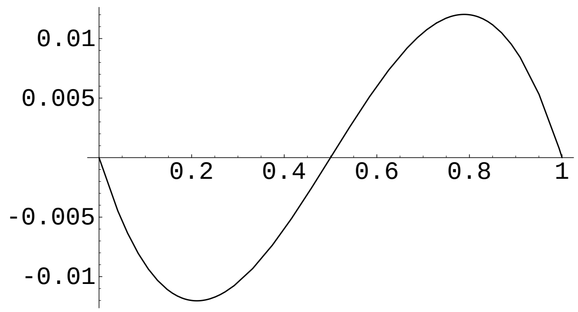
Non-intersection constraints

$$\nu_1 = 4\delta_1(\delta_0 - \beta_0 + \beta_2) - (\beta_1 - \beta_3 + \delta_2 + \delta_3)^2$$
$$\nu_2 = 4\delta_1(\delta_0 - \beta_0 - \beta_2) - (\beta_1 + \beta_3 - \delta_2 - \delta_3)^2$$
$$\nu_3 = 4\alpha_1(\alpha_0 - \gamma_0 + \gamma_2) - (\alpha_2 + \alpha_3 + \gamma_1 - \gamma_3)^2$$
$$\nu_4 = 4\alpha_1(\alpha_0 - \gamma_0 - \gamma_2) - (\alpha_2 + \alpha_3 - \gamma_1 - \gamma_3)^2$$

Must all be > 0

Direction of vector field





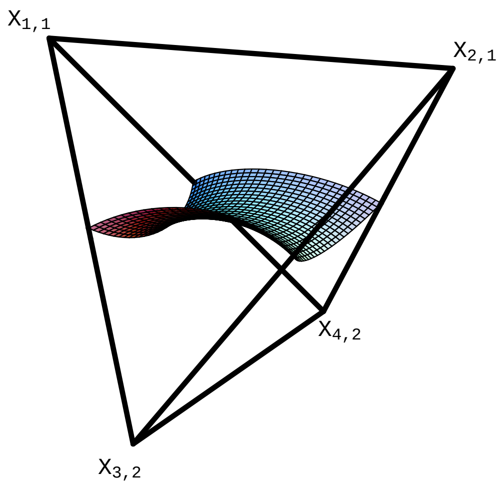
$$f(z) = \dot{z}|_{r=0,s=0}$$

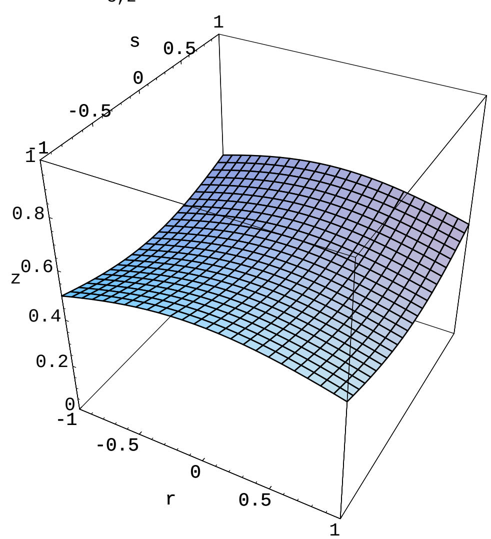
- Need $f'(0) < 0$ & $f'(1) < 0$:
  - $f'(0) = \dfrac{\beta_0 - \delta_0}{2}$
  - $f'(1) = \dfrac{\gamma_0 - \alpha_0}{2}$
- So assume:
  - $\delta_0 - \beta_0 > 0$
  - $\alpha_0 - \gamma_0 > 0$

Summary of this result

- Six inequalities:
  - $\nu_1 > 0, \nu_2 > 0, \nu_3 > 0, \nu_4 > 0$
  - $\delta_0 - \beta_0 > 0$
  - $\alpha_0 - \gamma_0 > 0$
- These imply:
  - interior null-cline between top and bottom
  - vector field upward above, downward below
  - top and bottom faces of box are attracting sets
  - top and bottom edges of simplex are attracting sets
  - means UGs are stable against invasion

Case: very different UGs

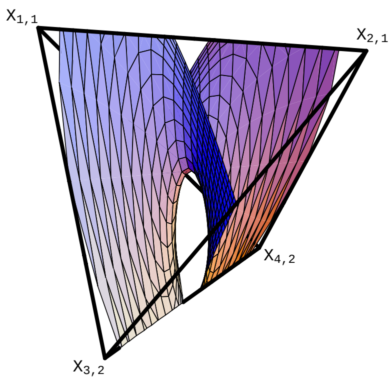


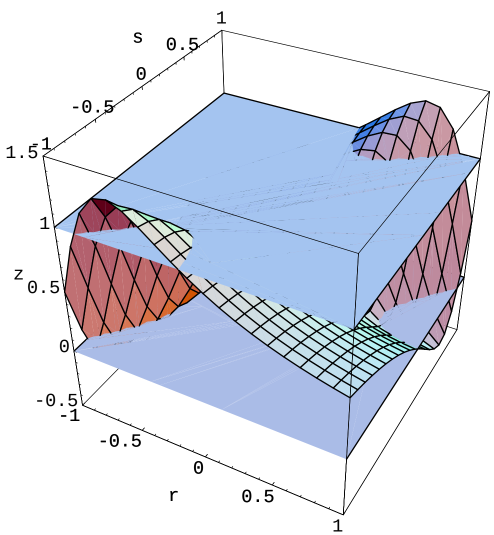


$$B = \begin{pmatrix} c & a & \varepsilon & \varepsilon \\ a & c & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & c & a \\ \varepsilon & \varepsilon & a & c \end{pmatrix}$$

- $c =$ big, $a =$ medium, $\varepsilon =$ small
- $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 4(c - a)(c + a - 2\varepsilon)$
- $\alpha_0 - \gamma_0 = \delta_0 - \beta_0 = a + c - 2\varepsilon$
- Generally stable

Case: similar UGs

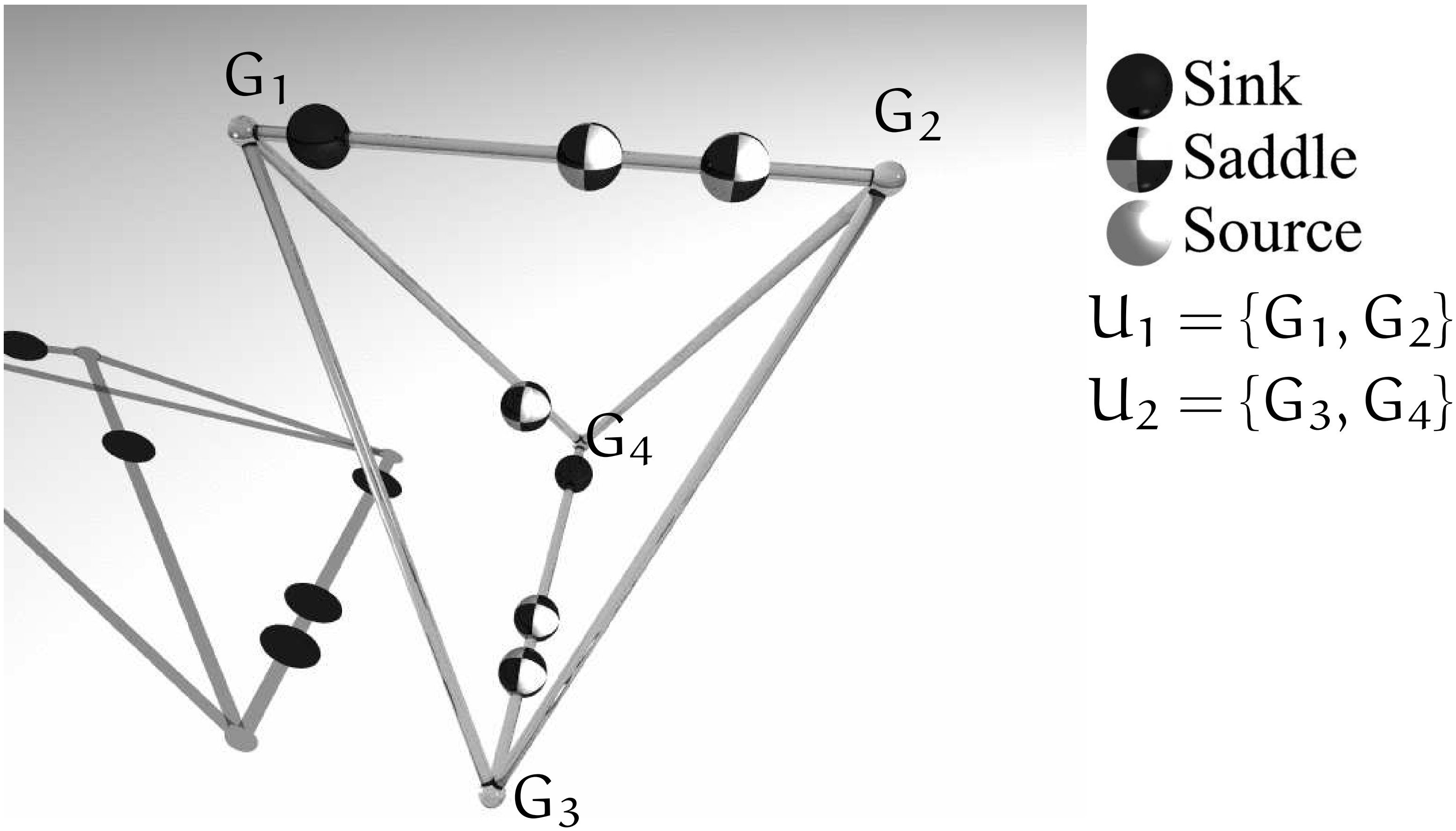




$$B = \begin{pmatrix} c & a & (1-\varepsilon)c & (1-\varepsilon)a \\ a & c & (1-\varepsilon)a & (1-\varepsilon)c \\ (1-\varepsilon)c & (1-\varepsilon)a & c & a \\ (1-\varepsilon)a & (1-\varepsilon)c & a & c \end{pmatrix}$$

- $c =$ big, $a =$ medium, $\varepsilon =$ small
- $\nu_1 = \nu_2 = \nu_3 = \nu_4 = -(a-c)^2 - 2(a^2 + 2ac - 3c^2)\varepsilon - (a-c)^2\varepsilon^2$
- $\alpha_0 - \gamma_0 = \delta_0 - \beta_0 = (a+c)\varepsilon$
- Unstable?

Accidental stability



$U_1 = \{G_1, G_2\}$

$U_2 = \{G_3, G_4\}$

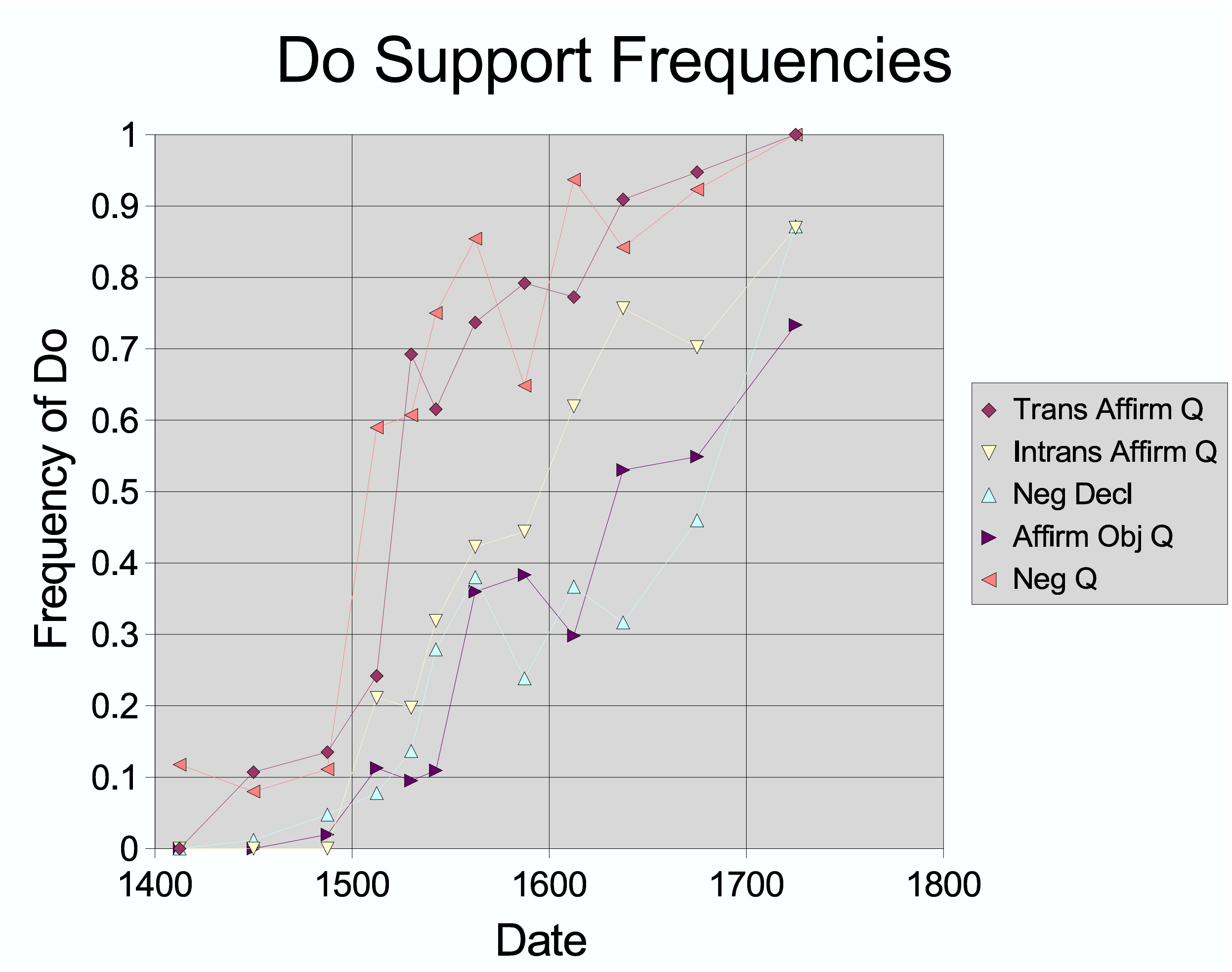
More on acquisition

- Child acquisition: short term
- Language change: long term

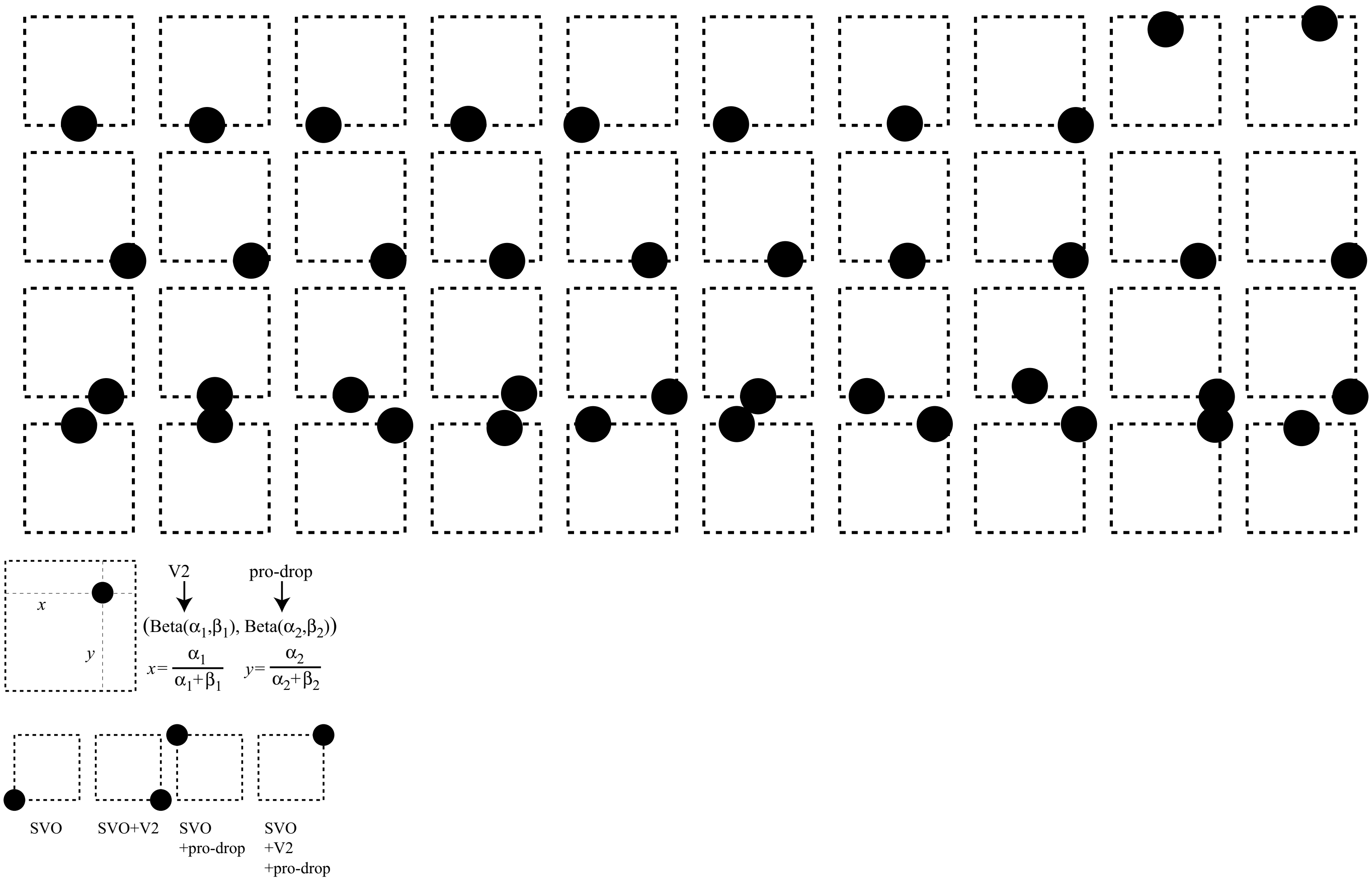
General requirements

- Medium term stability
- Long term variability
- Correct transitions
- Connect to data
- Specific Project: English
  - Lots of linguistic theory
  - Parsed corpora
  - Long interesting history

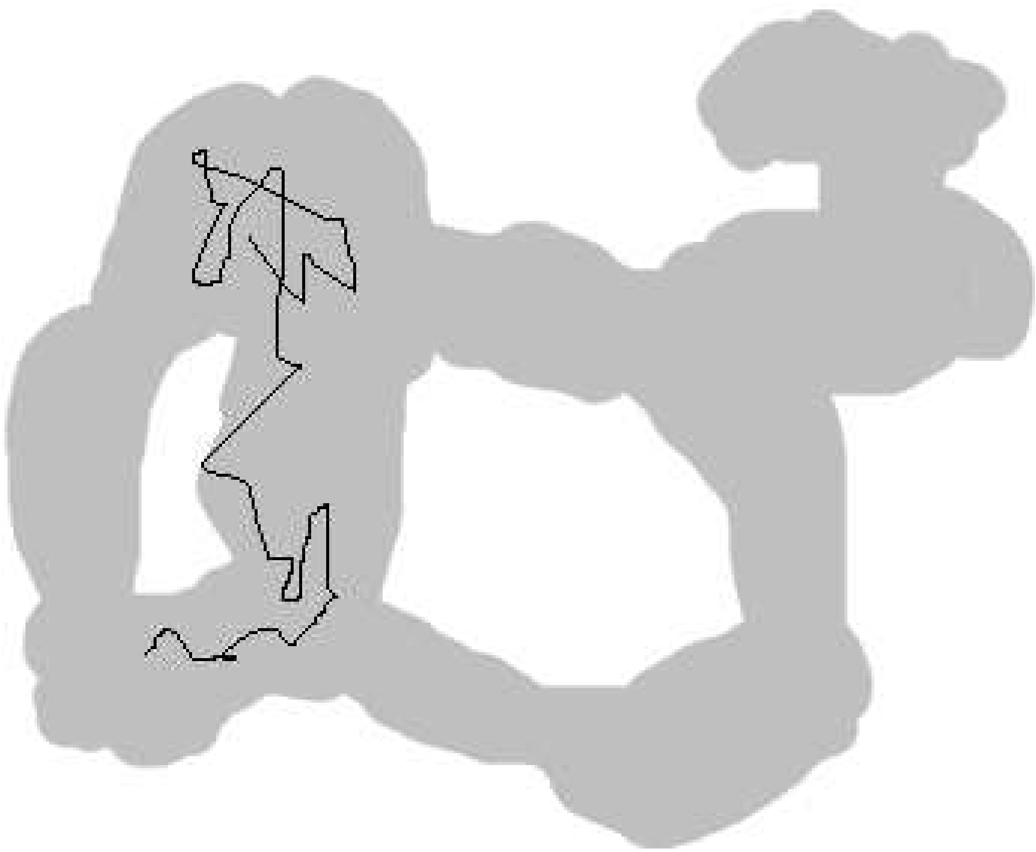
Mixtures of idealized grammars



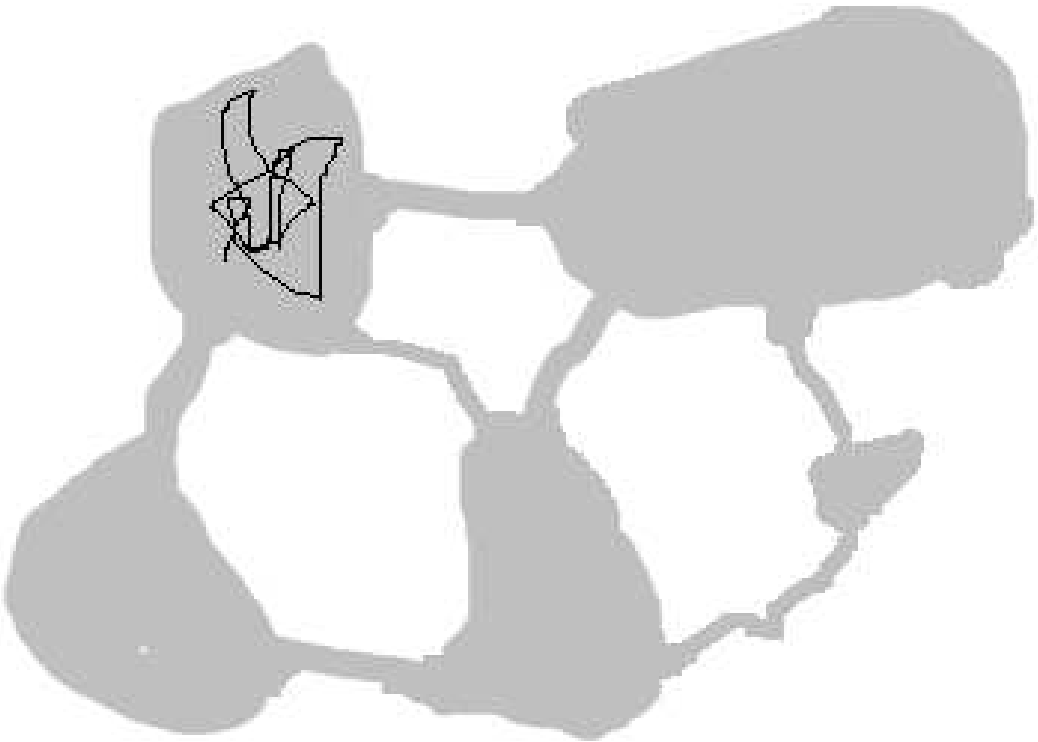
Learning and fuzzy grammars



Fast mixing Markov chain



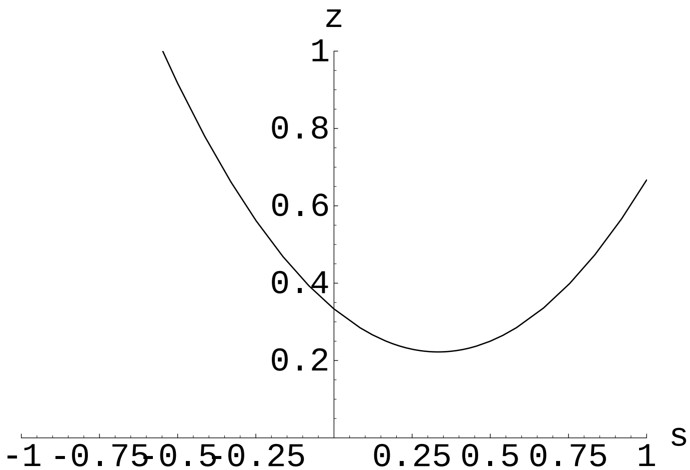
Slow mixing Markov chain



Papers & such

- W. G. M., Bifurcation Analysis of the Fully Symmetric Language Dynamical Equation. *J. M. Bio.*, v. 46 #3, March 2003.
- W. G. M. and M. A. Nowak, Competitive Exclusion and Coexistence of Universal Grammars. *Bull. M. Bio.*, v. 65 #1, January 2003.
- W. G. M. and Martin A. Nowak, Chaos and Language. *Proc. R. Soc. B*, v. 271 #1540, April 7 2004.
- `www.math.duke.edu/~wgm`

Sign of g_2



$g_2(1, s)$

$$y = x^2 + ax + b$$

$$\rightsquigarrow y_{\mathrm{min}} = b - \frac{a^2}{4}$$

$$\min_s g_2(1, s) = - \frac{-4(-\beta_0 + \beta_2 + \delta_0)\delta_1 + (\beta_1 - \beta_3 + \delta_2 + \delta_3)^2}{4\delta_1^2}$$

In between

$$g(0,0,\bar{z}) = 0$$

$$\rightsquigarrow \bar{z} = \frac{\delta_0 - \beta_0}{\alpha_0 - \gamma_0 + \delta_0 - \beta_0}$$

$$0 < \bar{z} < 1$$

$$\Leftarrow \delta_0 - \beta_0 > 0 \text{ and } \alpha_0 - \gamma_0 > 0$$