ON THE LATTICE OF EQUIVALENCE RELATIONS

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Permutable Equivalences

Equivalences and Partitions

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Theorem 1. *Let R and T be equivalences in a set S. The following statements are equivalent:*

(a) R *and* T *permute*

(b) $R \vee T = RT$

(c) *RT is an equivalence relation.*

Independence of Partitions

Independence in Information Theory

Definition. *Two partititions a and b are said to be* ***independent*** *if $\rho \cap \tau \neq \emptyset$ for all blocks $\rho \in a$ and $\tau \in b$. We also say that two equivalences are independent when they are determined by independent partititions.*

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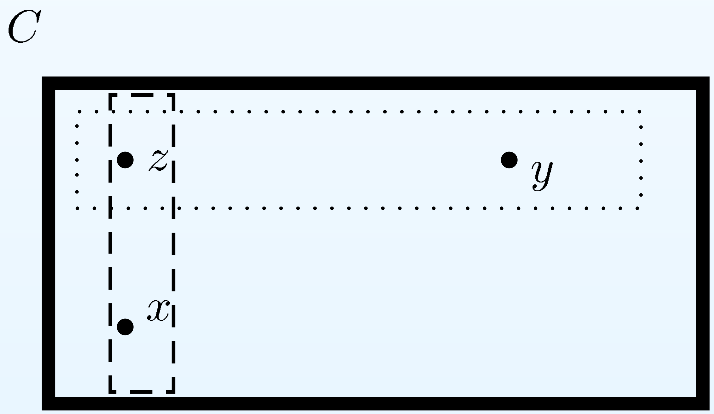

figure 1: Perspective of the equivalence $R_{a \vee b}$ 'internal' to each block.

Independence e Permmutability

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Two equivalences permute when the correspondent partititions are disjoint sums of independent partititions.

R-T Compatible Equivalences

R-T Compatibility

Definição. An equivalence relation E in S is said to be R-T compatible if the following conditions are satisfied:

$$(b_1) \quad \forall x,y,z \in S, xEyRz \Rightarrow [\,x\,]_R \cap [\,z\,]_T \cap [\,z\,]_E \neq \emptyset$$

$$(b_2) \quad \forall x,y,z \in S, xEyTz \Rightarrow [\,x\,]_T \cap [\,z\,]_R \cap [\,z\,]_E \neq \emptyset$$

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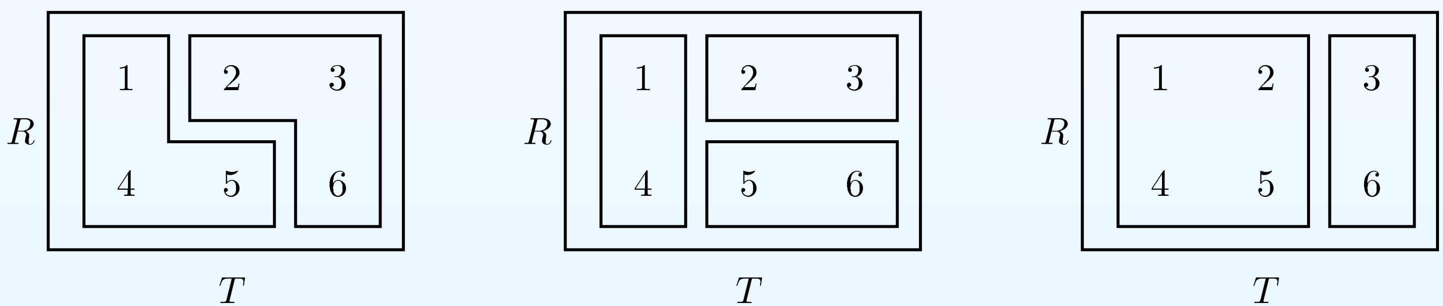

Equivalence E — Equivalence E’ — Equivalence E”

figure 2: Some examples of equivalence pairs.

A Caracterization for the R-T Compatibility

Consider, for each class C of S/E the following sets:

$$\begin{aligned} R_C &= \{\, \rho \in S/R : \rho \cap C \neq \emptyset \,\} \\ T_C &= \{\, \tau \in S/T : \tau \cap C \neq \emptyset \,\} \\ (R,T)_C &= \{\, (\rho,\tau) \in S/R \times S/T : \rho \cap \tau \cap C \neq \emptyset \,\} \end{aligned}$$

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Theorem 6. *Let S be a set and $E \in Eq(S)$ such that $R \cap T \subseteq E$. Then E is $R - T$ compatible if, and only if, the following conditions are satisfied:*

(1) $\forall C \in S/E,\ (R,T)_C = R_C \times T_C,$

(2) $\forall C, C' \in S/E,\ R_C, R'_C$ *are either disjoint or identical.*

(3) $\forall C, C' \in S/E,\ T_C, T'_C$ *are either disjoint or identical.*

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Theorem 7. *The family of $R - T$ compatible equivalences is a sublattice of $Eq(S)$ bouded by $R \vee T$ and $R \wedge T$.*

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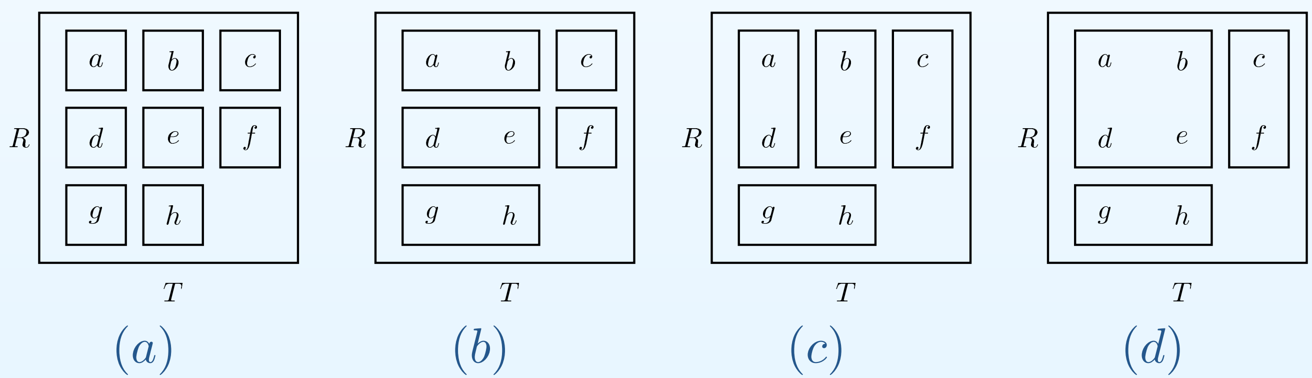

figure 3: Some examples of equivalence pairs in $S = \{a, b, c, d, e, f, g, h\}$ and (d) is R-T compatible.

The Shape of a Structure

Let $F = S/R * T$. Consider the relations R^F,T^F defined in F by

$$R^F = \{\, (C, C') \in F \times F : R_C \bigcap R_{C'} \neq \emptyset \,\}$$

[the blocks of $R * T$ that intersect in their lines]

$$T^F = \{\, (C, C') \in F \times F : T_C \bigcap T_{C'} \neq \emptyset \,\}$$

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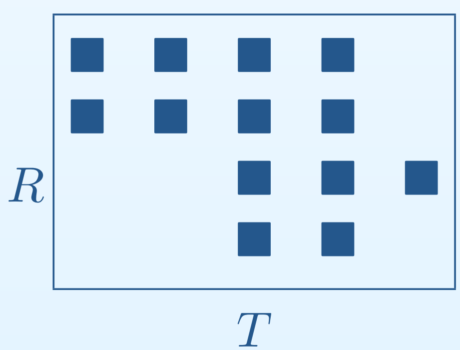
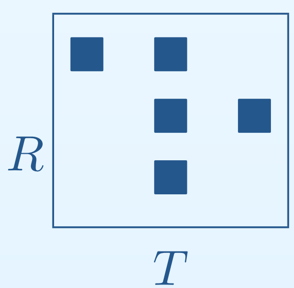
Definição. The **shape** of $(S; R, T)$ is the struture $(F; R^F, T^F)$.



Struture $(S; R, T)$



Shape $(F; R^F, T^F)$

From the Struture to the Shape

Consider the functions $f : F/R \rightarrow \wp(F/T)$ and $g : F/T \rightarrow \wp(F/R)$ defined by

$$f(\rho) = \{\, \tau \in F/T : \tau \cap \rho \neq \emptyset \,\} = T_\rho \text{ em } F\text{, para cada } \rho \in F/R$$
$$g(\tau) = \{\, \rho \in F/R : \rho \cap \tau \neq \emptyset \,\} = R_\tau \text{ em } F\text{, para cada } \tau \in F/T$$

T_ρ and R_τ are, respectively, the set of columns of the shape F that the line ρ intersects, and the set of lines that the column τ intersects.

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Proposition 8. *The following statements are equivalent:*

(a) $(F; R, T)$ *is a shape,*
(b) $R * T = \Delta_S$
(c) f *e* g *are injective.*

From the Shape back to the Struture

To get back the structure, through it's shape, having in mind that R^F and T^F are equivalences in $F \times F$, we consider the following functions:

$r : F/R \longrightarrow \mathbb{N}$, defined by $r(\rho) = |R_C|$, for some $C \in \rho$
$t : F/T \longrightarrow \mathbb{N}$, defined by $t(\tau) = |R_C|$, for some $C \in \tau$.

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Remark. *Every struture $(S; R, T)$ such that $R \wedge T = \Delta_S$ is determined, up to isomorphism, by:*

(i) *its shape $(F; R^F, T^F)$,*
(ii) *the function $r : F/R^F \longrightarrow \mathbb{N}$ and*
(iii) *the function $c : F/T^F \longrightarrow \mathbb{N}$.*

Fundamental Theorem of Equivalence Permmutability

Theorem 9. *Dubreil 1939*
Two equivalences R_a and R_b permmute if, and only if, for each block C of the partition $a \vee b$, the partitions $a_{|C}$ and $b_{|C}$ are independent.

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This is equivalent to the statement that $R \vee T$ is R-T compatible. In general, we may restate the fundamental theorem as follows:

Theorem 10. *The equivalences R and T permute if, and only if, $R \vee T = R * T$*

End of Presentation