A Stochastic Gradient Method with an Exponential Convergence Rate for Finite Training Sets

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Context : Machine Learning for "Big Data"

- **Large-scale machine learning** : large n, large p
  - n : number of observations (inputs)
  - p : number of parameters in the model
- **Examples** : vision, bioinformatics, speech, language, etc.
  - Pascal large-scale datasets : $n = 5 \cdot 10^5, p = 10^3$
  - ImageNet : $n = 10^7$
  - Industrial datasets : $n > 10^8, p > 10^7$
- Main computational challenge :
  - Design algorithms for very large n and p.

A standard machine learning optimization problem

We want to minimize the sum of a finite set of smooth functions :

$$\min_{\theta \in \mathbb{R}^p} g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta)$$

For instance, we may have

$$f_i(\theta) = \log\left(1 + \exp\left(-y_i x_i^\top \theta\right)\right) + \frac{\lambda}{2} \|\theta\|^2$$

We will focus on strongly-convex functions g.

Deterministic methods

$$\min_{\theta \in \mathbb{R}^p} g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta)$$

Gradient descent updates

$$\begin{aligned} \theta_{k+1} &= \theta_k - \alpha_k g'(\theta_k) \\ &= \theta_k - \frac{\alpha_k}{n} \sum_{i=1}^{n} f_i'(\theta_k) \end{aligned}$$

- Iteration cost in $O(n)$
- Linear convergence rate $O(C^k)$
- Fancier methods exist but still in $O(n)$

Stochastic methods

$$\min_{\theta \in \mathbb{R}^p} g(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta)$$

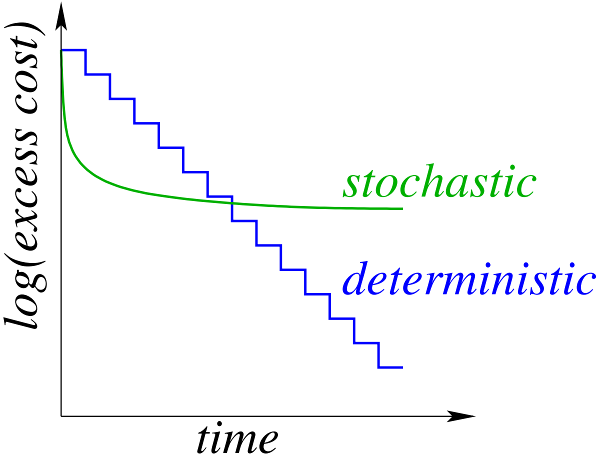
Stochastic gradient descent updates

$$i(k) \sim \mathcal{U}[\![1, n]\!]$$

$$\theta_{k+1} = \theta_k - \alpha_k f'_{i(k)}(\theta_k)$$

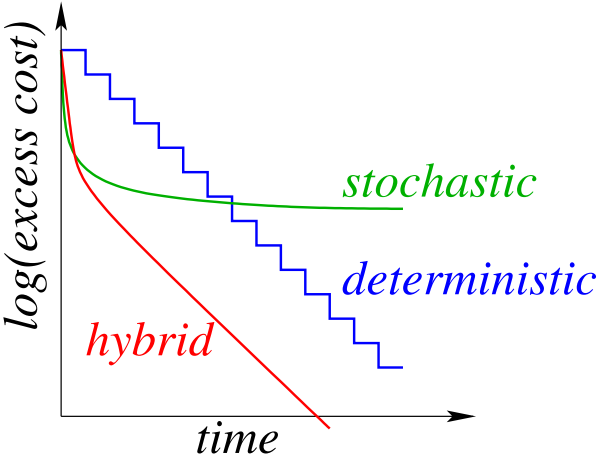
- Iteration cost in $O(1)$
- Sublinear convergence rate $O(1/k)$
- Bound on the test error valid for one pass

Hybrid methods



Hybrid methods

Goal = linear rate **and** $O(1)$ iteration cost.



Related work - Sublinear convergence rate

- **Stochastic version of full gradient methods**
  - Schraudolph (1999), Sunehag et al. (2009), Ghadimi and Lan (2010), Martens (2010), Xiao (2010)
- **Momentum, gradient/iterate averaging**
  - Polyak and Judistky (1992), Tseng (1998), Nesterov (2009), Xiao (2010), Kushner and Yin (2003), Hazan and Kale (2011), Rakhlin et al. (2012)
- None of these methods improve on the $O(1/k)$ rate

Related work - Linear convergence rate

- **Constant step-size SG, accelerated SG**
  - Kesten (1958), Delyon and Juditsky (1993), Nedic and Bertsekas (2000)
  - Linear convergence but only up to a fixed tolerance
- **Hybrid methods, incremental average gradient**
  - Bertsekas (1997), Blatt et al. (2007), Friedlander and Schmidt (2012)
  - Linear rate but iterations make full passes through the data
- **Stochastic methods in the dual**
  - Shalev-Shwartz and Zhang (2012)
  - Linear rate but limited choice for the f_i's

Stochastic Average Gradient Method

Full gradient update :

$$\theta_{k+1} = \theta_k - \frac{\alpha_k}{n} \sum_{i=1}^{n} f_i'(\theta_k)$$

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Stochastic Average Gradient Method

Stochastic average gradient update :

$$\theta_{k+1} = \theta_k - \frac{\alpha_k}{n} \sum_{i=1}^{n} y_i^k$$

- **Memory** : $y_i^k = f'_{i(k')}(\theta_{k'})$ from the last k' where i was selected.
- Random selection of $i(k)$ from $\{1, 2, \ldots, n\}$.
- Only evaluates $f'_{i(k)}(\theta_k)$ on each iteration.

Stochastic variant of incremental average gradient [Blatt et al., 2007]

SAG convergence analysis

- Assume each f_i' is L-continuous, average g is μ-strongly convex.
- With step size $\alpha_k \leqslant \frac{1}{2nL}$, SAG has linear convergence rate.
- **Linear convergence with iteration cost independent of** n.
- With step size $\alpha_k = \frac{1}{2n\mu}$, if $n \geqslant 8\frac{L}{\mu}$ then

$$\mathbb{E}[g(\theta_k) - g(\theta_*)] \leqslant C\left(1 - \frac{1}{8n}\right)^k.$$

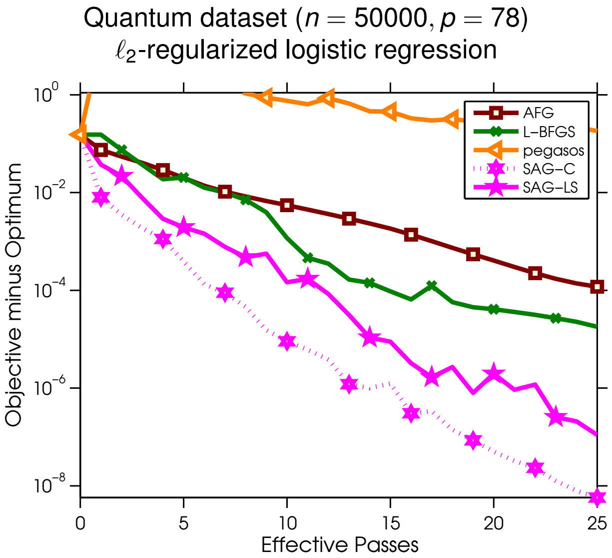
- Rate is “independent” of the condition number.
  - Constant error reduction after each pass,

$$\left(1 - \frac{1}{8n}\right)^n \leq \exp\left(-\frac{1}{8}\right) = 0.8825.$$

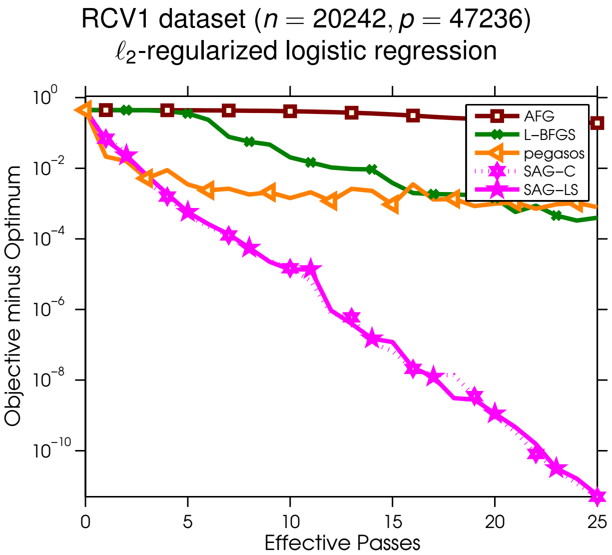
Comparison with full gradient methods

- Assume $L = 100$, $\mu = 0.01$ and $n = 80000$:
  - Full gradient has rate $\left(1 - \frac{\mu}{L}\right)^2 = 0.9998$
  - Accelerated gradient has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.9900$
  - SAG (n iterations) multiplies the error by $\left(1 - \frac{1}{8n}\right)^n = 0.8825$
  - Fastest possible first-order method has rate $\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.9608$
- We beat two lower bounds (with additional assumptions)
  - Stochastic gradient bound
  - Full gradient bound

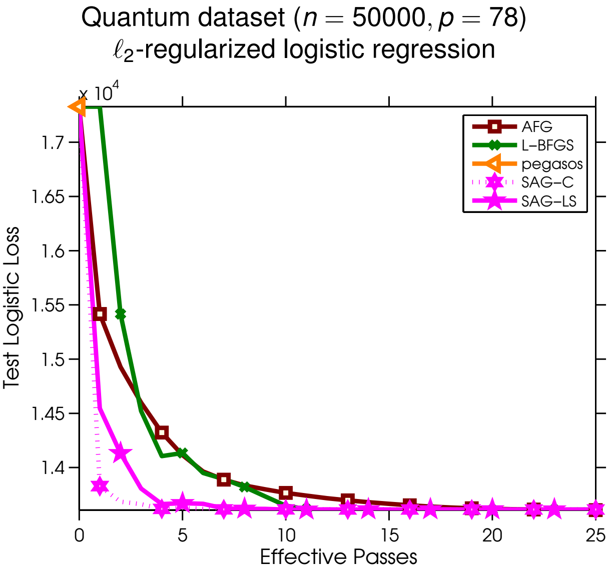
Experiments - Training cost



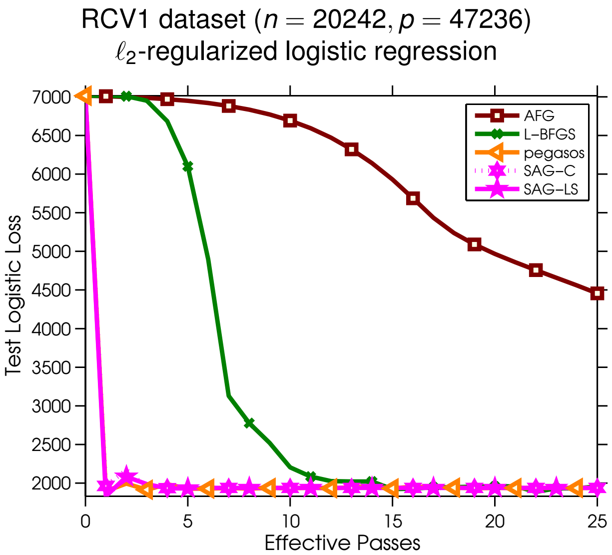
Experiments - Training cost



Experiments - Testing cost



Experiments - Testing cost



Reducing memory requirements

- $\theta_{k+1} = \theta_k - \frac{\alpha_k}{n} \sum_{i=1}^{n} y_i^k$
- y_i^k is the last gradient computed on datapoint i
- Memory requirement : $O(np)$
- Smaller for structured models, e.g., linear models :
  - If $f_i(\theta) = \ell(y_i, x_i^\top \theta)$, $f_i'(\theta) = \ell'(y_i, x_i^\top \theta) x_i$
  - Memory requirement : $O(n)$
- We can also use mini-batches

Conclusion and Open Problems

- Fast theoretical convergence using the ‘sum’ structure common in applications.
- Simple algorithm, empirically better than theory predicts.
- Allows line-search and approximate optimality measures.
- Open problems :
  - Large-scale distributed implementation.
  - Determine a tight convergence rate in all cases.
  - Apply the method to constrained and non-smooth problems.
  - Speed up the method using non-uniform sampling and non-Euclidean metric.