MAKING CENTRALIZED (GRAPH) COMPUTATION FASTER, DISTRIBUTED AND (AT TIMES) BETTER

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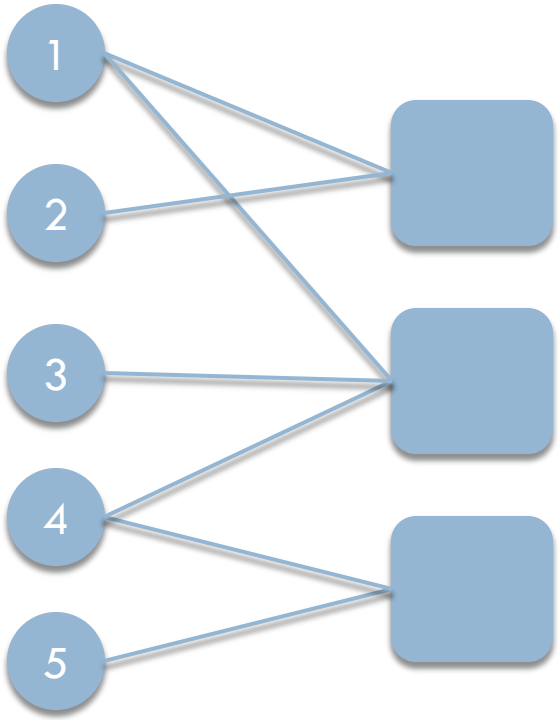
Graphical models

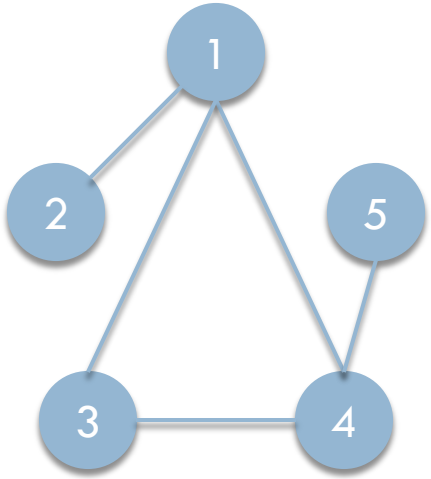
- Graphs provide canonical representation
  - To capture "interaction" of all sorts

- Examples
  - Social communications:
    - Who (e-)talks to whom
  - Cascading dynamics:
    - Failure in e-grid
  - Mutual (in)dependence:
    - Price of oil and political (in)stability

Example

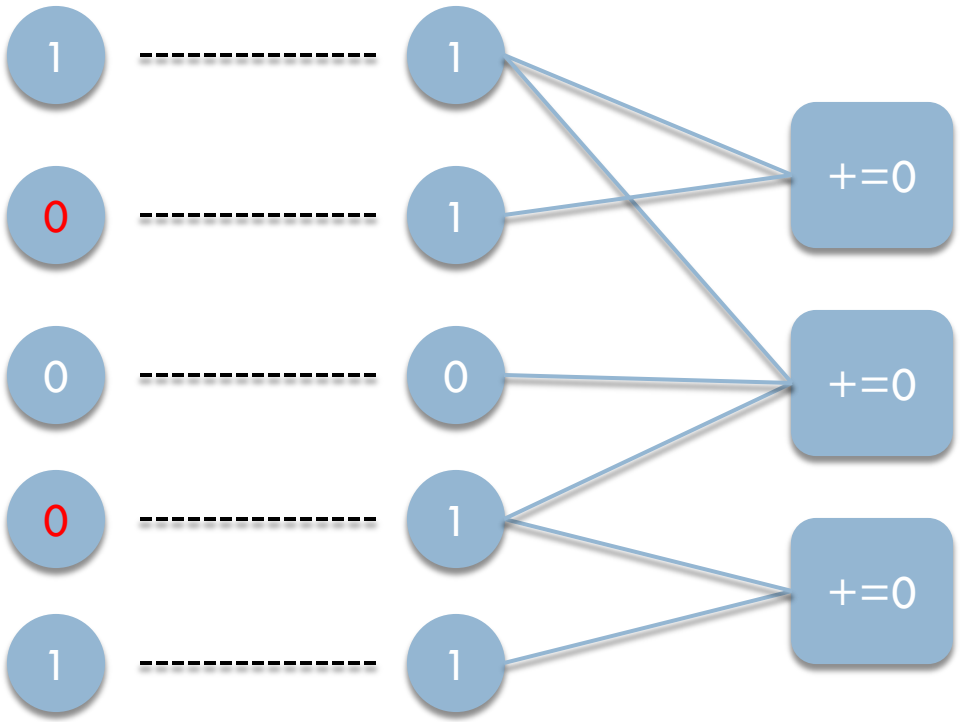


Ground Truth



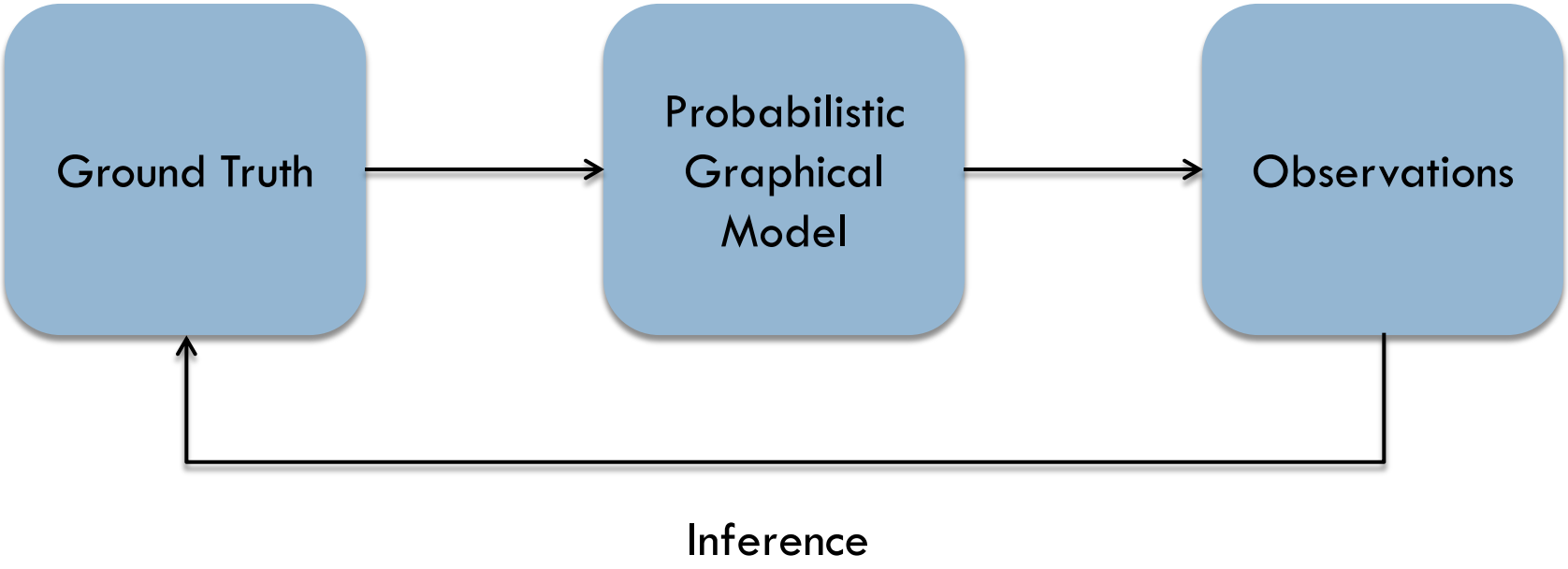
Observations

Example



Observations Ground Truth

Probabilistic model



- Maximum A Posteriori/Maximum likelihood estimation
  - A graph optimization problem

Inference: Clustering

- Newmann '06
  - Connections between nodes inside a cluster are less *likely to resemble randomly formed connections*

    (and likely to be more than random)

- Modularity metric:
  - Graph G=(V,E) with V={1,…,n}
    - d_i be degree of node i,
    - A_{ij} is 1 if (i,j) is an edge in E
  - Modularity of a subset C of V is

$$\mathbf{M}(C) = \sum_{ij \in C} \left(A_{ij} - \frac{d_i d_j}{2|E|} \right)$$

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  - *Most likely* clustering optimizes overall modularity: choose clusters $C_1,\ldots,C_k$ so that they

$$\text{maximize} \sum_l \mathbf{M}(C_l) \text{ over } \textit{all possible choices}$$

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$$\text{maximize} \sum_{ij \in V} \mathrm{I}(\chi(i) = \chi(j)) \left(A_{ij} - \frac{\mathrm{d_i d_j}}{2|\mathrm{E}|} \right) \text{ over } \chi : \mathrm{V} \rightarrow \{1,...,\mathrm{n}\}$$

    - *Computationally hard (to approximate)*

Inference: pair-wise graphical model

- Formally
  - Graph G = (V, E)
    - V = {1,…,N} correspond to N variables, say $X_1,\ldots,X_N$
      - Each variable takes value in finite alphabets, say {0,1}
    - E is set edges capturing "dependencies"
  - Joint distribution:

$$P(X_1 = x_1,\ldots, X_N = x_N) \propto \exp\left(\sum_{i\in V} \phi_i(x_i) + \sum_{(i,j)\in E} \psi_{ij}(x_i, x_j)\right)$$

    - Node potentials: ϕ_i for all $i \in V$
    - Edge potentials: ψ_{ij} for all $(i,j) \in E$

Inference: pair-wise graphical model

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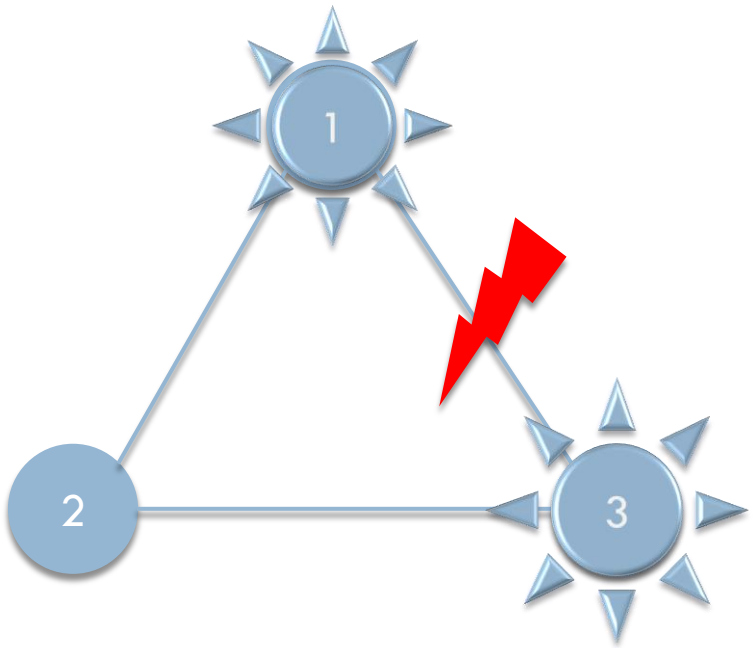
  - Potentials: weighted independent set model
    - $\phi_i(\sigma) = w_i \sigma$ for $\sigma \in \{0,1\}$ for all $i \in V$
    - $\psi_{ij}(\sigma, \sigma') = -\infty$ if $(\sigma, \sigma') = (1,1)$ and 0 o.w. for all $(i,j) \in E$

Inference: pair-wise graphical model



- Scheduling by sampling as per this distribution is capacity achieving when node potentials/weights = func(queue-size)
  - This view has led to totally distributed Medium Access [Shah-Shin '08 '11]

Inference: pair-wise graphical model

- Find Maximum A Posteriori (MAP) assignment
  - x^* such that

$$x^* \in \operatorname{argmax} \sum_{i \in V} \phi_i(x_i) + \sum_{(i,j) \in E} \psi_{ij}(x_i, x_j)$$

  - In the context of independent set model
    - Solve for maximum weight independent set

$$\begin{aligned} &\text{maximize} && \sum_{i \in V} w_i x_i \\ &\text{such that} && x_i + x_j \leq 1 \text{ for all } (i,j) \in E \\ &\text{over} && x_i \in \{0,1\} \text{ for all } i \in V \end{aligned}$$

    - Computationally hard (to approximate)

Optimization: "prominent" methods

- All optimization methods are highly relevant
  - Linear programming
  - Semi-definite programming
  - Higher-order relaxations
    - Lovasz-Schrijver
    - Sherali-Adams
    - Lassare and Sums-of-squares
  - (Approximate) Dynamic programming
  - …

Graphical model: "prominent" methods

- Graphical model induced techniques
  - Variational inference
    - Mean-field
    - Belief propagation
    - Tree-reweighted and its variations
  - Markov Chain Monte Carlo
    - Gibbs sampling
    - Metropolis-Hastings
    - Simulated annealing
  - …

This talk

- A method for solving graph optimization problems
  - Utilizes graph partitioning
    - Cf. Lipton-Tarjan '80, Baker '94; popularized recently in dist comp
  - Requires nearly linear time computation
  - Makes existing centralized heuristic distributed
- For a large class of graphs (poly-growth, planar, …)
  - Provides good, provably approximation
  - Speeds up computation of existing heuristics
    - Without losing performance
- For any graph, nearly linear time heuristic
  - With easy to evaluate error bound

Algorithm

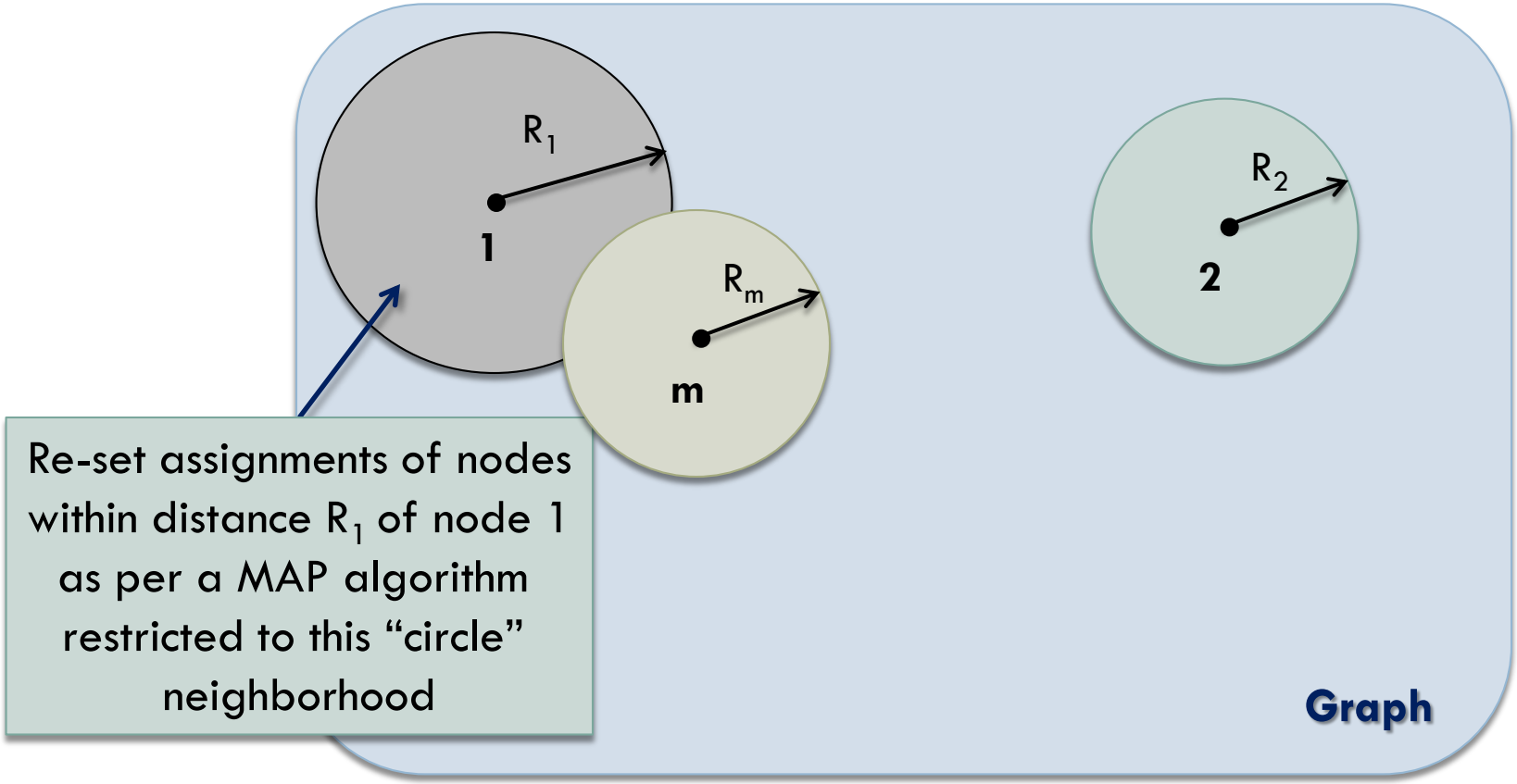
- Two parameters: K and δ

- Useful random variable R
  - Truncated Geometric(ε) with value at most K

$$\mathrm{P}(\mathrm{R}=\ell)=\begin{cases}\delta(1-\delta)^{\ell-1} & \ell<\mathrm{K}\\ (1-\delta)^{\mathrm{K}-1} & \ell=\mathrm{K}\end{cases}$$

Algorithm

- Initialize $X_i = 0$ for all i (or any valid assignment)
- Iteratively, algorithm updates the assignments



Algorithm

- Effectively,
  - Each iteration improves the assignment
    - Restricted to a small random (local) neighborhood
    - By means of possibly exact MAP (exp(K) computation)

- **Theorem 1 (Jung-Shah '08)** Upon selecting all N vertices once (in any order), the algorithms finds (1- ε) MAP approximation for graph with poly-growth rate ρ as long as K scales as $\Theta\left(\frac{\rho}{\varepsilon}\ln\frac{\rho}{\varepsilon}\right)$ (on average, whp upon log N repeats).

Graph with poly-growth

- We call a graph with poly-growth rate ρ if
  - For any vertex v of G and any r
    - N(r,v), number of nodes within graph hops r, is bounded as

$$N(r, v) \leq C\, r^{\rho}$$

for some constant C > 0.

- Example
  - Grid graph, or
  - *Any local* graph betn nodes in a finite dim'l metric space

Algorithm: general graph

- For any graph
  - Exact MAP inside local neighborhood
  - Let E' be set of edges "cut" by the N random partitions
  - Let $C(E') = \sum_{(i,j) \in E'} |\psi_{ij}|_\infty$

- **Lemma 1 (Jung-Shah '08)** Upon selecting all N vertices once (in any order), the algorithms finds MAP approximation whose cost is at most C(E') less than exact MAP for any graph.

Randomized decomposition

- Under our randomized decomposition
  - For graph with poly-growth
  - (and with appropriate choice of K, δ)

Pr(an edge in "cut") ~ε

  - This leads to the desired approximation bound

Algorithm for modularity opt

- Effectively,
  - Each iteration improves the assignment
    - Restricted to a small random (local) neighborhood
    - By means of possibly exact Modularity (super-exp(K) computation)

- **Theorem 2 (Blondel-Jung-Shah '11)** Upon selecting all N vertices once (in any order), the algorithms finds clustering that is within (1- ε) of maximal Modularity for graph with poly-growth rate ρ as long as K scales as $\Theta\left(\frac{\rho}{\varepsilon}\ln\frac{\rho}{\varepsilon}\right)$ (on average, whp upon log N repeats)..

Is it an excellent result?

- Well, looks like it…
  - We've got a linear time PTAS
- But, can we *really* use it
  - Constants scale super-exponentially in $1/\varepsilon$
  - On a 100 node graph
    - DP for ind. set requires $2^{100} \cong 10^{30}$ operations!
    - Modularity opt requires 100^{100} operations!
- Okay, the usual *fantastic* algorithmic result

Algorithm: a *useful* variation

- Use of approximate MAP
  - Suppose inside each "local" neighborhood, we use
    - An approximation MAP algorithm (e.g. Belief propagation)
    - Let it be A(N) MAP approximation (for any graph)
      - with $A(N) = o(N)$ (e.g. $A(N) = \log N$)

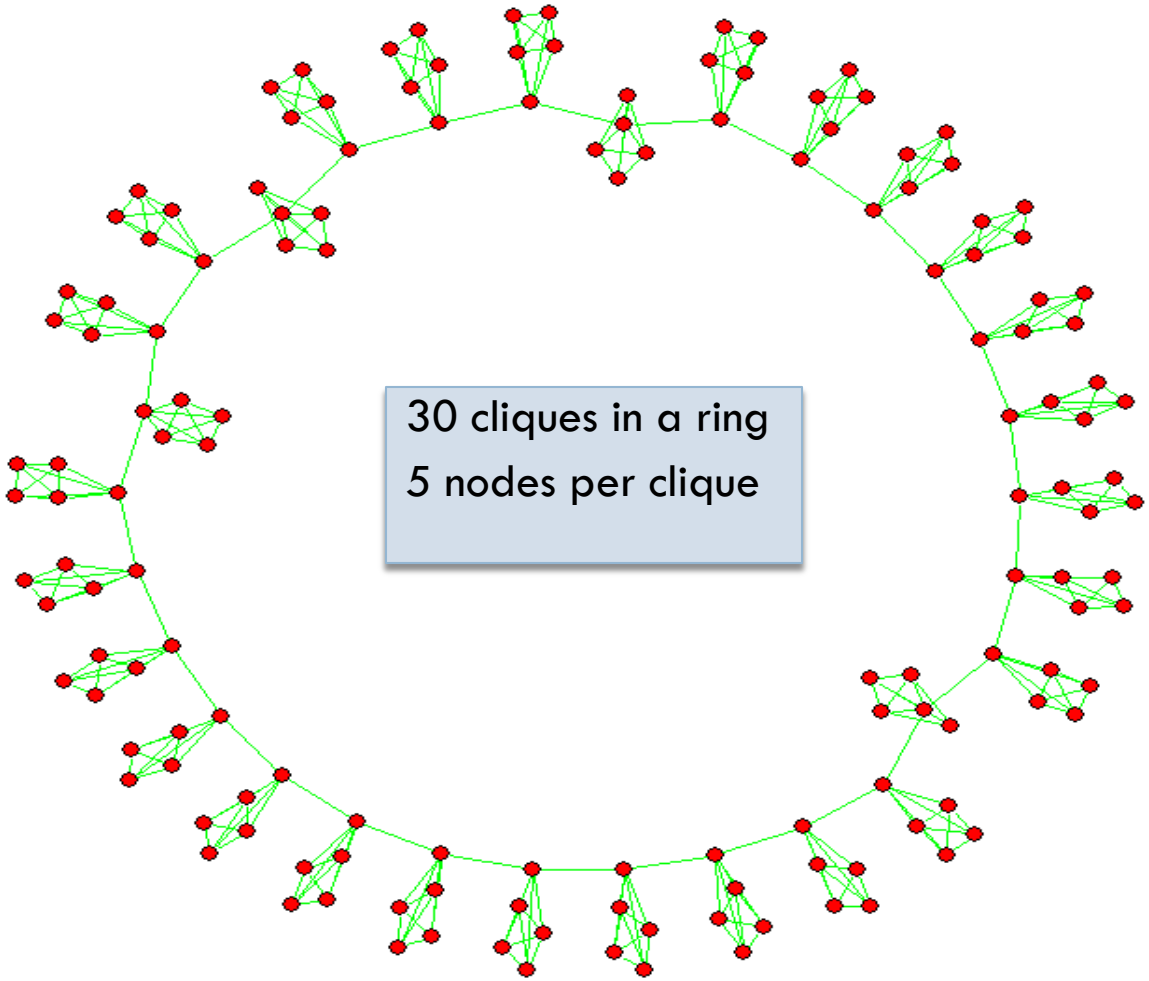
- **Theorem 3 (Jung-Shah '11)** Upon selecting all N vertices once (in any order), the algorithms finds *constant factor* MAP approximation for graph with poly-growth rate ρ as long as K scales as before.

Algorithm: a *useful* variation

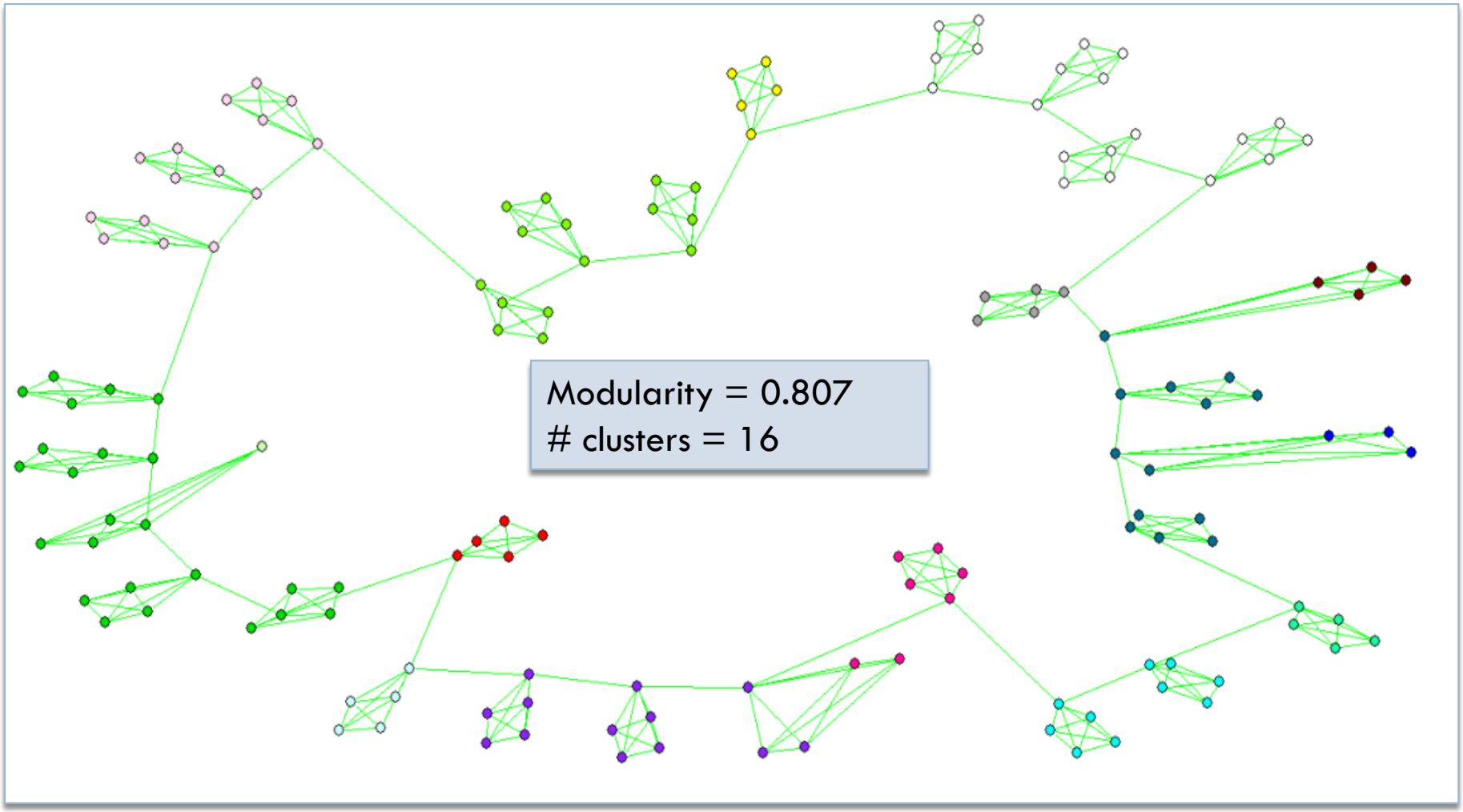
- Use of approximate MAP
  - Suppose inside each "local" neighborhood, we use
    - An approximation Mod Opt algorithm (e.g. Blondel et al '09)
    - Let it be A(N) Mod Opt approximation (for any graph)
      - with $A(N) = o(N)$ (e.g. $A(N) = \log N$)

- **Theorem 4 (Blondel-Jung-Shah '11)** Upon selecting all N vertices once (in any order), the algorithms finds *constant factor* Mod Opt approximation for graph with poly-growth rate ρ as long as K scales as before.

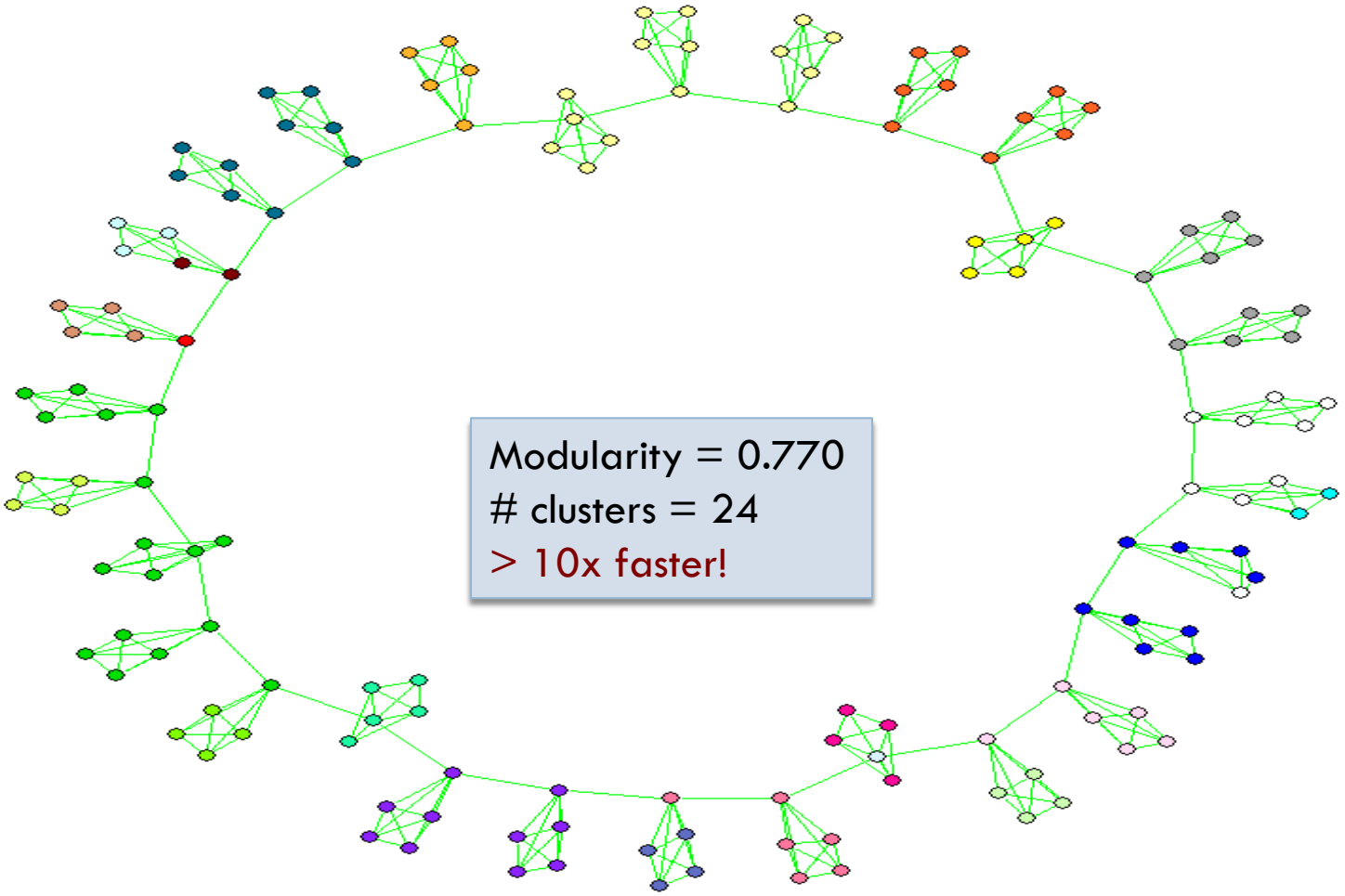
Example



Example: Blondel et al



Example: Blondel et al + partition



Algorithm: variations

- Local property testing (Sudan-Widgerson '98)
  - Does a given graph have certain property P
  - By means of an algorithm that has
    - only error dependent randomized computation cost

- Every monotone graph property is locally testable
  - Alon and Shapira '05
  - Constant: "towering exponential" dependence on $1/\varepsilon$

Algorithm: variations

- Partitioning can be used for property testing
  - Hassidim, Kelner, Nguyen, and Onak '10
  - Reduces the towering dependence to one exponent
  - Homogenous properties

- This algorithm can be used as
  - Local property tester for graphs with poly-growth
  - For various "optimization"-based properties
    - That are not necessarily homogenous

Algorithm: variations

- Algorithm can be used for
  - Estimating log partition function
    - Or normalization constant
  - With guarantees similar to those for MAP
  - It works as a "proof" to establish existence of limit
    - For normalized free-energy
      - for poly-growth graphs with limiting "regular" structure
    - And hence, limit can be evaluated

Summary

- "Meta" algorithm for inference in graphical model
  - For excellent approximation for graphs with geometry
  - Speeds up existing heuristics without losing performance
  - Has easy to evaluate "error" bound for any graph
  - It is effectively the architecture of choice

- Other useful consequences
  - Log partition function (learning)
  - Property testing