Blackwell Approachability <==> No-Regret Online Learning

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David Blackwell
1919 - 2010

- Born April 24, 1919
- Earned PhD at age 22, was 7th black PhD in math in US
- Professor at Howard Univ., traditionally black school, 1946-1955
- Move to Stat. dept. at UC Berkeley in 1955
- First black tenured prof at Berkeley

REVIEW: Online Linear Optimization

Online Linear Optimization

Decision Space: $\mathcal{K} \overset{\text{cvx}}{\subset} \mathbb{R}^d$ compact, convex

For $t = 1, \ldots, T$:

- Player chooses $\mathbf{x}_t \in \mathcal{K}$
- Adversary chooses linear (convex) $\mathbf{f}_t(\cdot)$
- Player suffers $\mathbf{f}_t(\mathbf{x}_t)$ (or, $\mathbf{f}_t \cdot \mathbf{x}_t$)

Regret: $$\sum_{t=1}^{T} \mathbf{f}_t(\mathbf{x}_t) - \min_{\mathbf{x}^* \in \mathcal{K}} \sum_{t=1}^{T} \mathbf{f}_t(\mathbf{x}^*)$$

Problem Solved...

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Algorithms
Online Gradient Descent
Follow the Perturbed Leader
Weighted Majority
Exponentiated Gradient
Follow the Regularized Leader
....

Rates

Setting	Regret rate
$\mathcal{K} := \Delta_d, \mathbf{f}_t \in [0,1]^d$	$\sqrt{T \log d}$
$\mathcal{K} := B_2, \mathbf{f}_t \in B_2$	$\sqrt{T}$
$\mathcal{K} := [0,1]^d, \mathbf{f}_t \in [0,1]^d$	$d\sqrt{T}$
$\mathcal{K} := B_2, \mathbf{f}_t \in [0,1]^d$	$\sqrt{dT}$
$\nabla \mathbf{f}_t \in B_2, \nabla^2 \mathbf{f}_t \succeq \alpha I$	$\frac{\log T}{\alpha}$

Key: rate is always o(T)

Blackwell Approachability

Warmup: Von Neumann Minimax Theorem

$$\max_{p \in \Delta_n} \min_{q \in \Delta_m} p^\top M q = \min_{q \in \Delta_m} \max_{p \in \Delta_n} p^\top M q$$

- Game matrix M -- M(i,j) is the payoff to P1 and the loss to P2
- $p^T Mq$ is the expected payoff when P1 plays p and P2 plays q
- Von Neumann: it doesn't matter which player goes first!

Von Neumann Ver. 2.0

Convex Compact Action Sets: $\mathcal{X} \overset{\text{cvx}}{\subset} \mathbb{R}^n, \mathcal{Y} \overset{\text{cvx}}{\subset} \mathbb{R}^m$

Biaffine Loss Function: $\ell(\mathbf{x}, \mathbf{y}) \in \mathbb{R}.$

If $\forall \mathbf{x} \in \mathcal{X} \; \exists \mathbf{y} \in \mathcal{Y} : \quad \ell(\mathbf{x}, \mathbf{y}) \in [c, \infty)$

Then $\exists \mathbf{y} \in \mathcal{Y} \; \forall \mathbf{x} \in \mathcal{X} : \quad \ell(\mathbf{x}, \mathbf{y}) \in [c, \infty)$

Blackwell’s question:

What if we’re paid in several currencies?
(E.G. dollars and happiness.)

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$$\textbf{Biaffine Loss Function:} \quad \ell(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^d.$$

$$\text{If } \forall \mathbf{x} \in \mathcal{X} \; \exists \mathbf{y} \in \mathcal{Y} : \quad \ell(\mathbf{x}, \mathbf{y}) \in S \subset \mathbb{R}^d$$

$$\text{Then } \exists \mathbf{y} \in \mathcal{Y} \; \forall \mathbf{x} \in \mathcal{X} : \quad \ell(\mathbf{x}, \mathbf{y}) \in S \subset \mathbb{R}^d$$

And an arbitrary convex "goal" set

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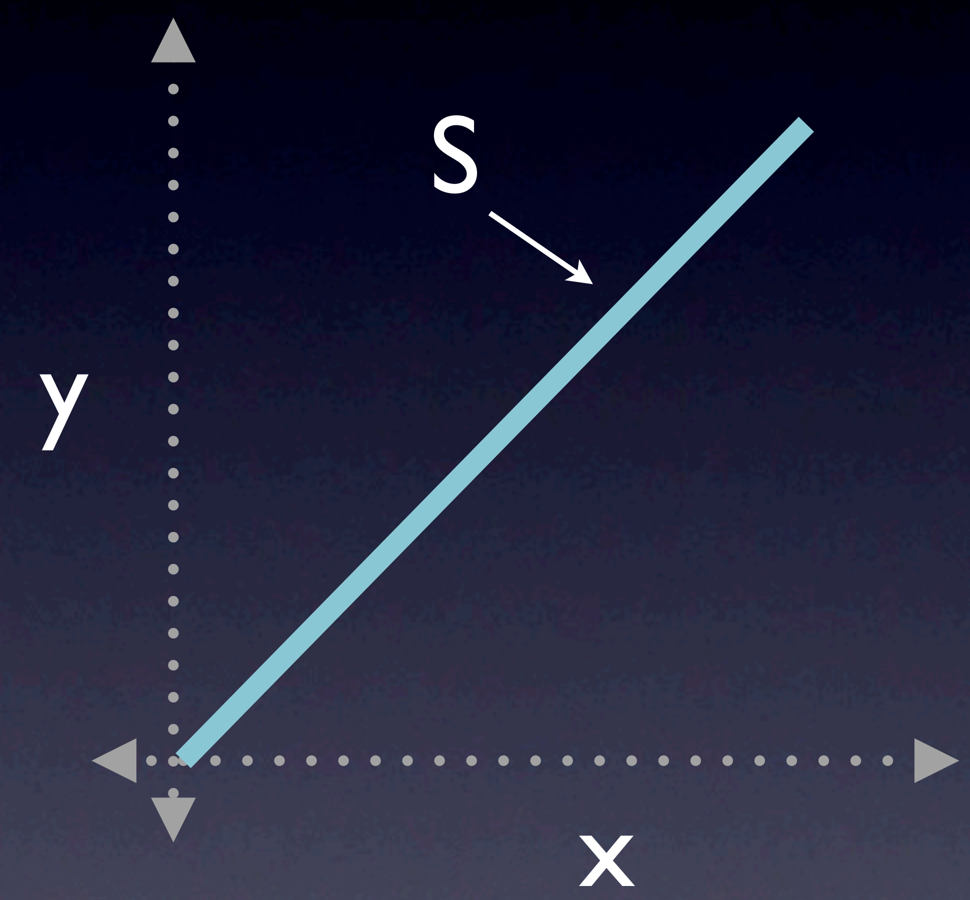
And an arbitrary convex "goal" set

Does this result still hold?

NOOOOOOOOO!!!!!

$$\mathcal{X} = \mathcal{Y} \quad = \quad [0,1]$$
$$\ell(x,y) \quad = \quad (x,y)$$
$$S = \{(z,z) \quad : \quad z \in [0,1]\}$$



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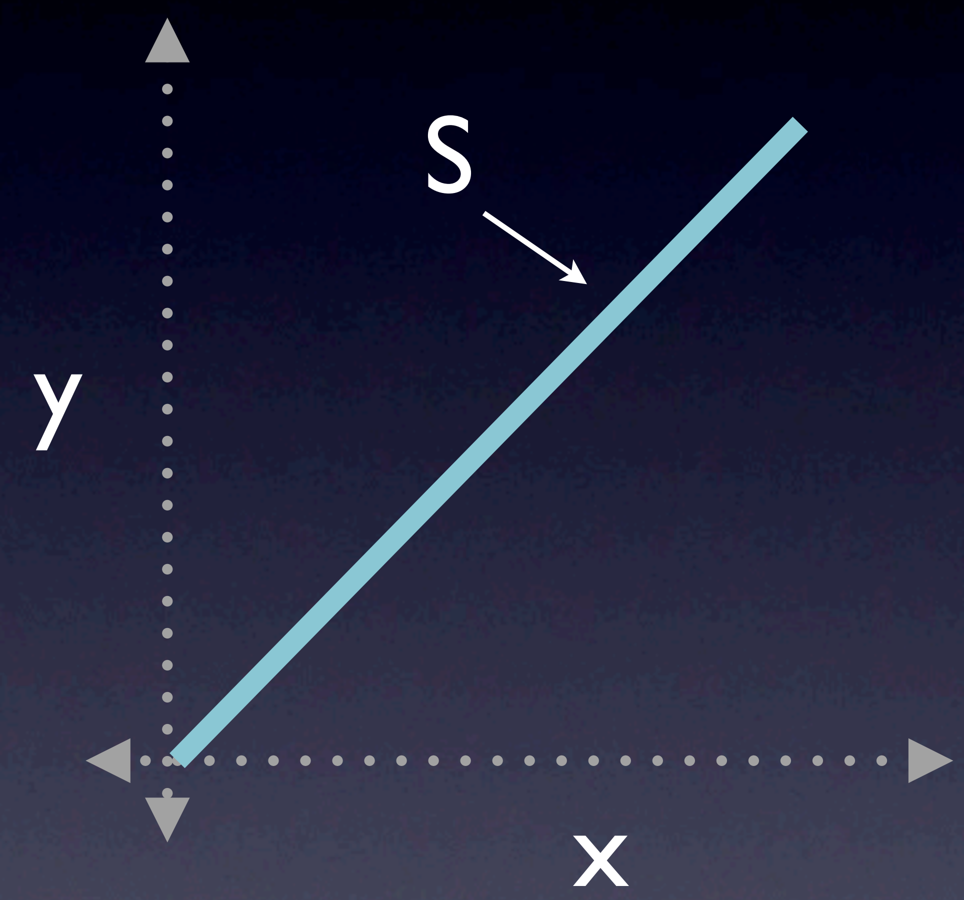
$$\forall x \exists y : (x,y) \in S?$$



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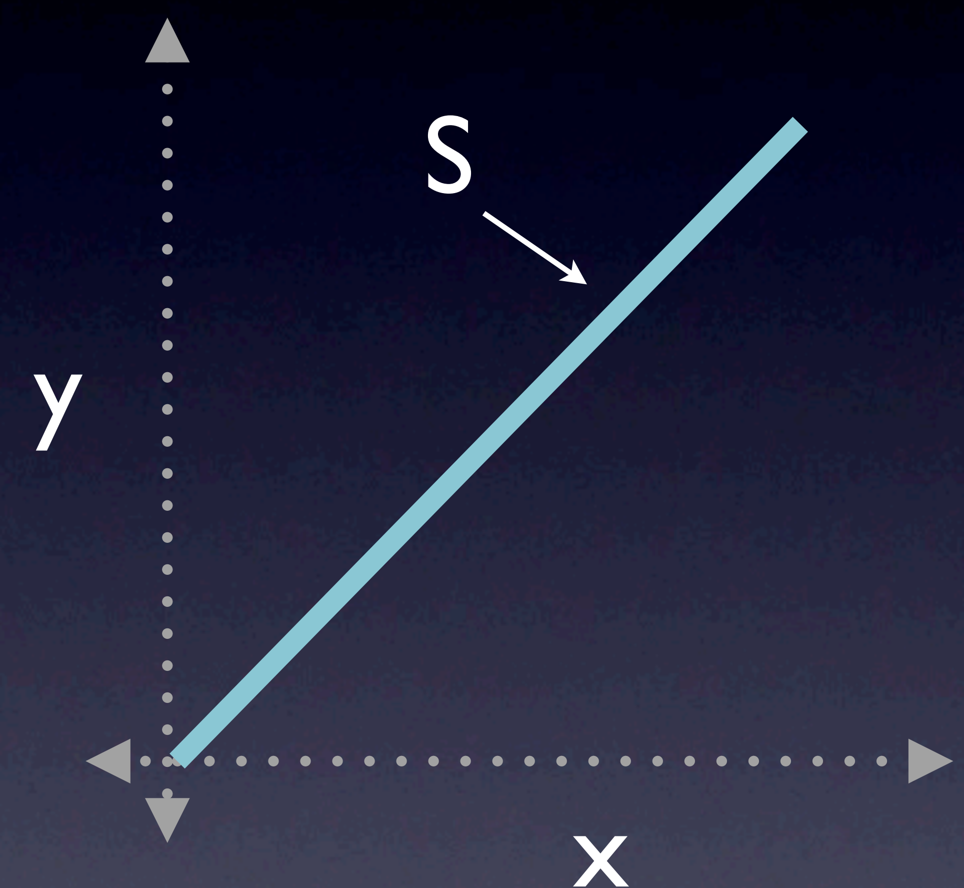

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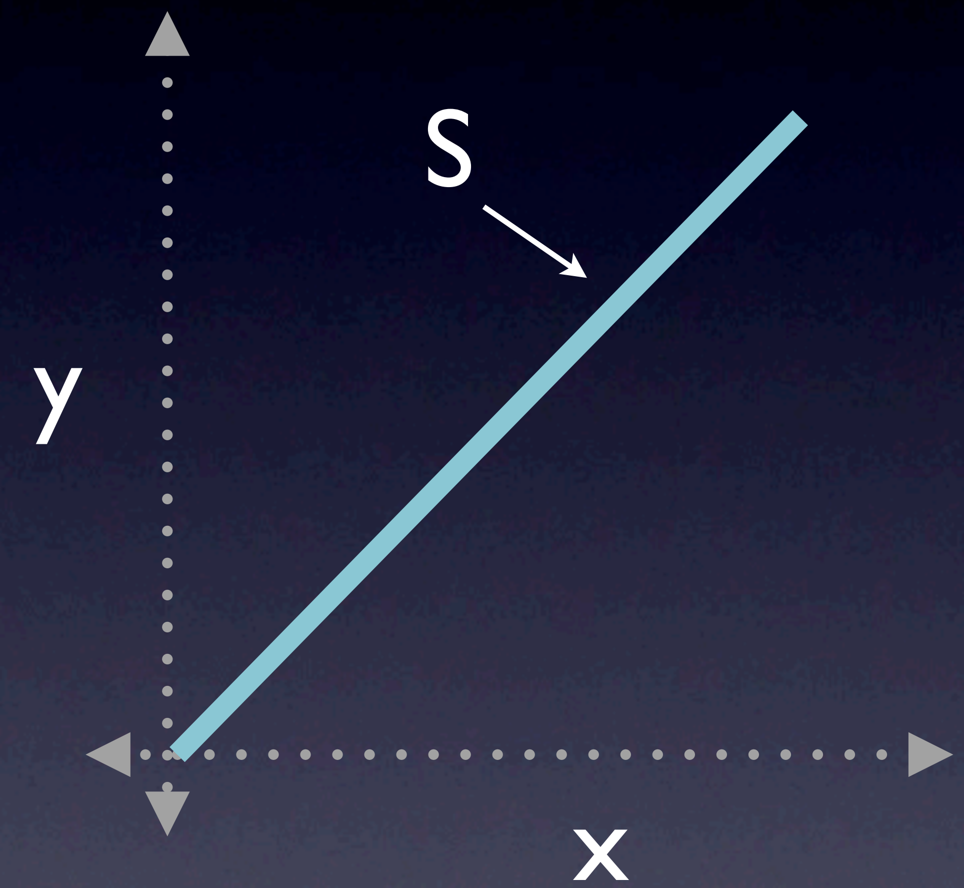

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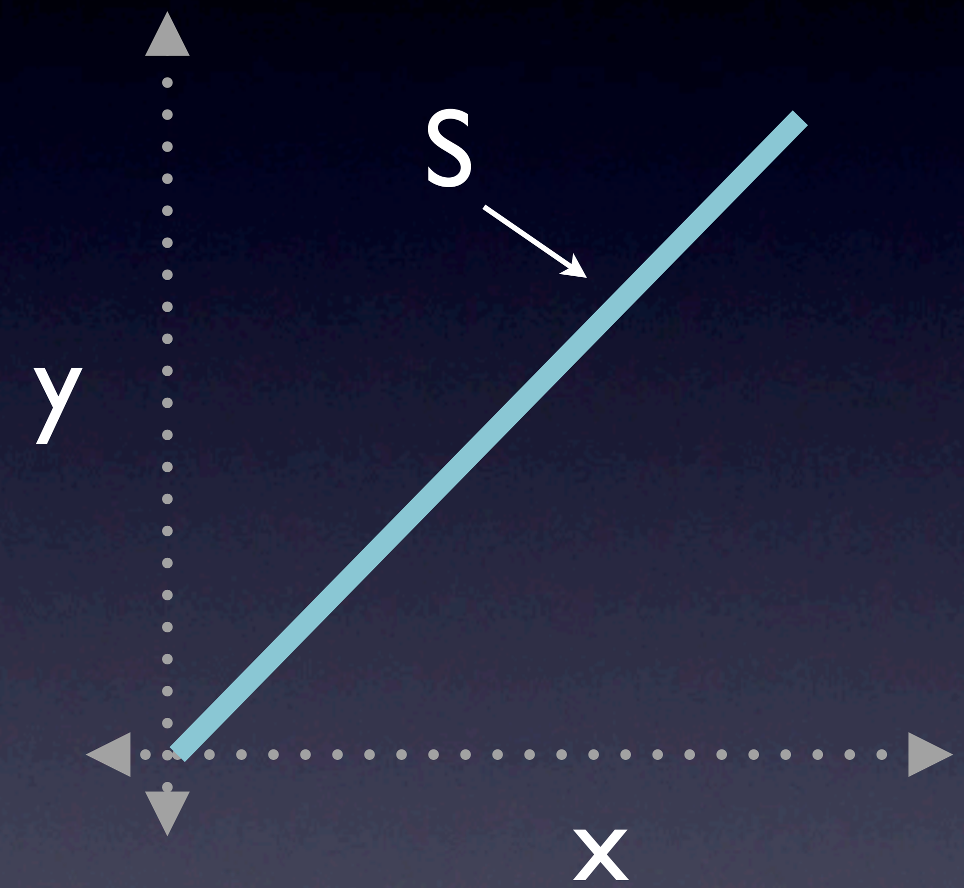

$\forall x \exists y : \ (x, y) \in S?$ ✓

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(womp womp...)

Definitions

S **Response-Satisfiable:** $\forall \mathbf{y}\ \exists \mathbf{x} : \ \ell(\mathbf{x}, \mathbf{y}) \in S$

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Definitions

S **Response-Satisfiable:** $\forall \mathbf{y}\ \exists \mathbf{x} : \ \ell(\mathbf{x}, \mathbf{y}) \in S$

$$\frac{1}{T}\sum_{t=1}^{T} \ell(\mathbf{x}_t, \mathbf{y}_t)$$

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$$\mathrm{dist}\left(\frac{1}{T}\sum_{t=1}^{T} \ell(\mathbf{x}_t, \mathbf{y}_t), S\right)$$

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$$\mathrm{dist}\left(\frac{1}{T}\sum_{t=1}^{T} \ell(\mathbf{x}_t, \mathbf{y}_t), S\right) \to 0$$

where $\mathbf{x}_t \leftarrow A(\mathbf{y}_1, \ldots, \mathbf{y}_{t-1})$

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Blackwell Approachability Theorem:
If S is response-satisfiable, then it is approachable

Old Example:

$$\mathcal{X} = \mathcal{Y} \quad = \quad [0,1]$$
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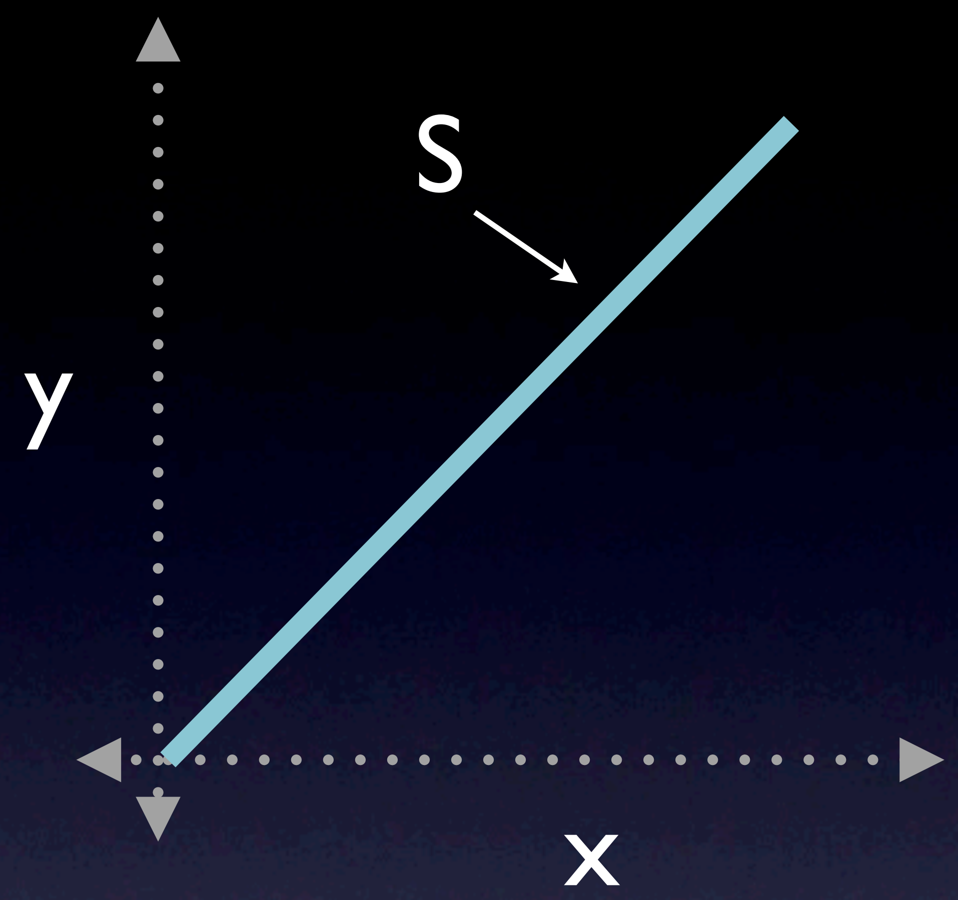

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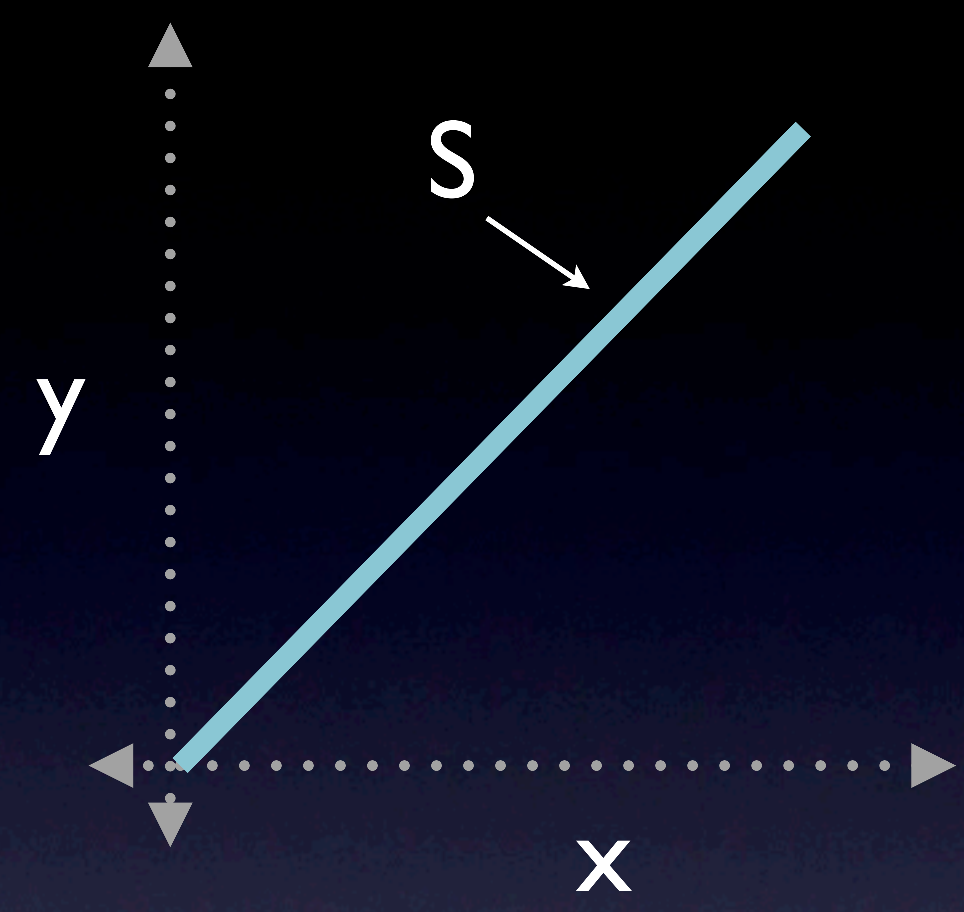

Need an alg. A s.t. $\forall y_1, y_2, \ldots \in [0,1]$:

$$\left(\frac{1}{T}\sum_t x_t, \frac{1}{T}\sum_t y_t\right) \to \{(z,z) : z \in [0,1]\}$$

where $x_t \leftarrow A(y_1, \ldots, y_{t-1})$.

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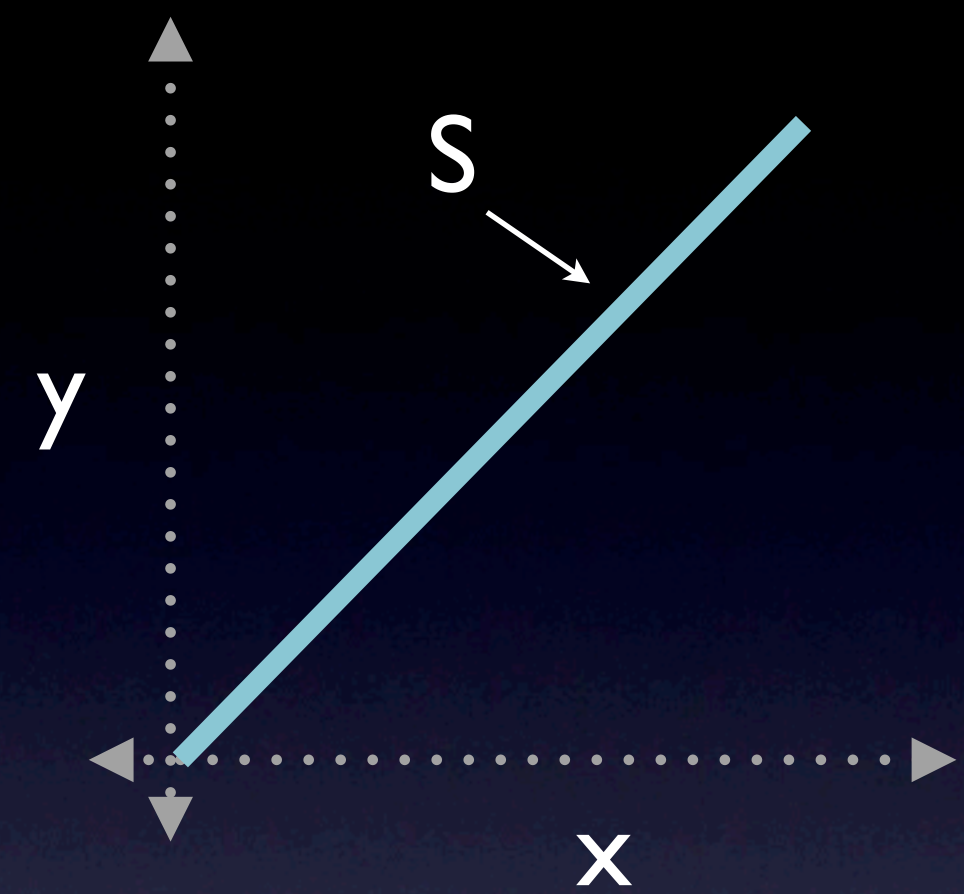

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where $x_t \leftarrow A(y_1, \ldots, y_{t-1})$.

EASY:
Simply set $x_t \leftarrow y_{t-1}$!!

Uses of Approachability (a very small sample)

- Empirical Bayes Stock Market Portfolios (Thomas M. Cover and David H. Gluss)
- Online Learning for Global Cost Functions. (E. Even-Dar, R. Kleinberg, S. Mannor, and Y. Mansour.)
- A Proof of Calibration via Blackwell's Approachability Theorem. (D. Foster)
- A Simple Adaptive Procedure Leading to Correlated Equilibrium. (Sergiu Hart and Andreu Mas-Colell)
- Calibration and Internal No-regret with Random Signals. (V. Perchet)
- On Repeated Games with Complete Information. (Sylvain Sorin)
- ....

Our Main Result

Minimax
Duality

$\wedge$

Regret
Minimization

$\wedge$

Blackwell
Approachability

Our Main Result

Minimax
Duality

^

Blackwell
Approachability

<==>

Regret
Minimization

OLO Regret Minimization

Given $\mathcal{K} \subset \mathbb{R}^d$, construct alg L so that $\forall \mathbf{f}_1, \mathbf{f}_2, \ldots \in \mathbb{R}^d$ we have

$$\sum_t \langle \mathbf{f}_t, \mathbf{x}_t \rangle - \min_{\mathbf{x} \in K} \sum_t \langle \mathbf{f}_t, \mathbf{x} \rangle = o(T),$$

where $\mathbf{x}_t \leftarrow L(\mathbf{f}_1, \ldots, \mathbf{f}_{t-1})$.

Blackwell Approachability

Given vector payoff $\ell : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^d$ construct an alg A so that $\forall \mathbf{y}_1, \mathbf{y}_2, \ldots \in \mathcal{Y}$,

$$\text{dist}\left(\tfrac{1}{T} \textstyle\sum_{t=1}^{T} \ell(\mathbf{x}_t, \mathbf{y}_t), S\right) \to 0$$

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NoRegret => Blackwell

Blackwell Problem

Input: cvx *cone* S, seq. of opp. actions $\mathbf{y}_1, \mathbf{y}_t, \ldots$

Guarantee: $\forall (H \supset S)\ \exists \mathbf{x}(H)\ \forall \mathbf{y}\ \ell(\mathbf{x}(H), \mathbf{y}) \in H$

Want: seq. $\mathbf{x}_1, \mathbf{x}_2, \ldots$ so that

$$\operatorname{dist}\left(\frac{1}{T}\sum_{t=1}^{T} \ell(\mathbf{x}_t, \mathbf{y}_t), S\right) = o(1)$$

OLO Construction

Define: $\mathcal{K} := \operatorname{dual}(S) \cap \operatorname{UnitBall}$

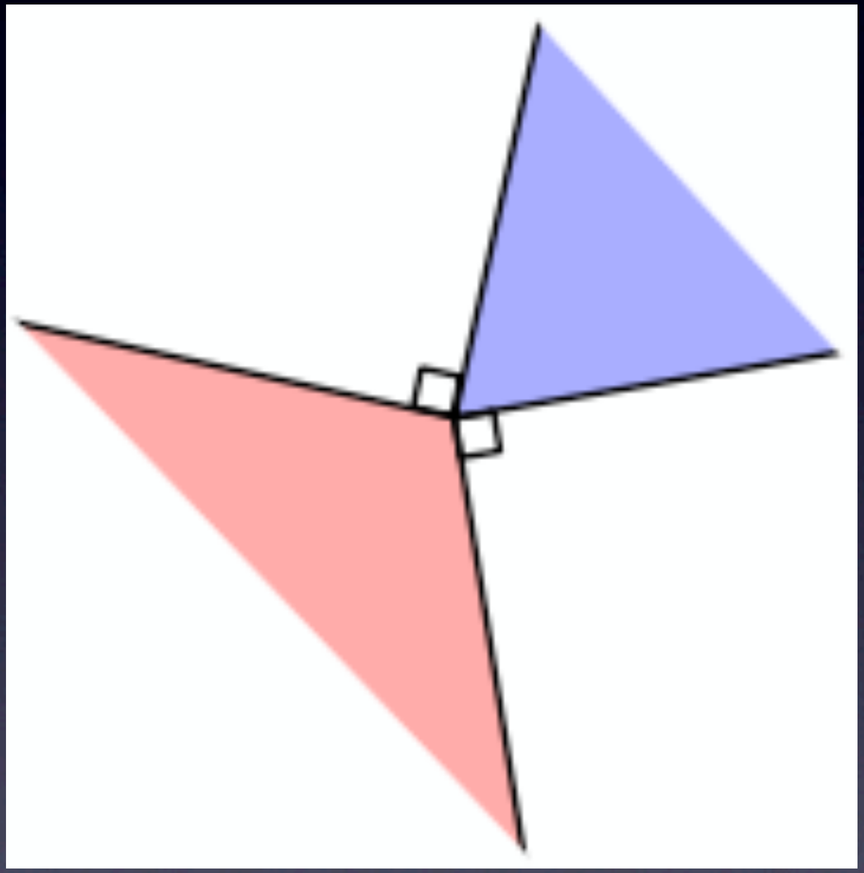
Define: $\mathbf{f}_t(\mathbf{h}) := \ell(\mathbf{x}_t, \mathbf{y}_t) \cdot \mathbf{h}$

Learn: $\mathbf{h}_1, \mathbf{h}_2, \ldots \in \mathcal{K}$

Choose: $\mathbf{x}_t \leftarrow \mathbf{x}(H_{\mathbf{h}_t})$

Trick: Cone Duality

$$C \text{ a cone } \implies C^0 := \{v \in \mathbb{R}^d : v \cdot w \leq 0 \; \forall w \in C\}$$



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Learning/
optimization
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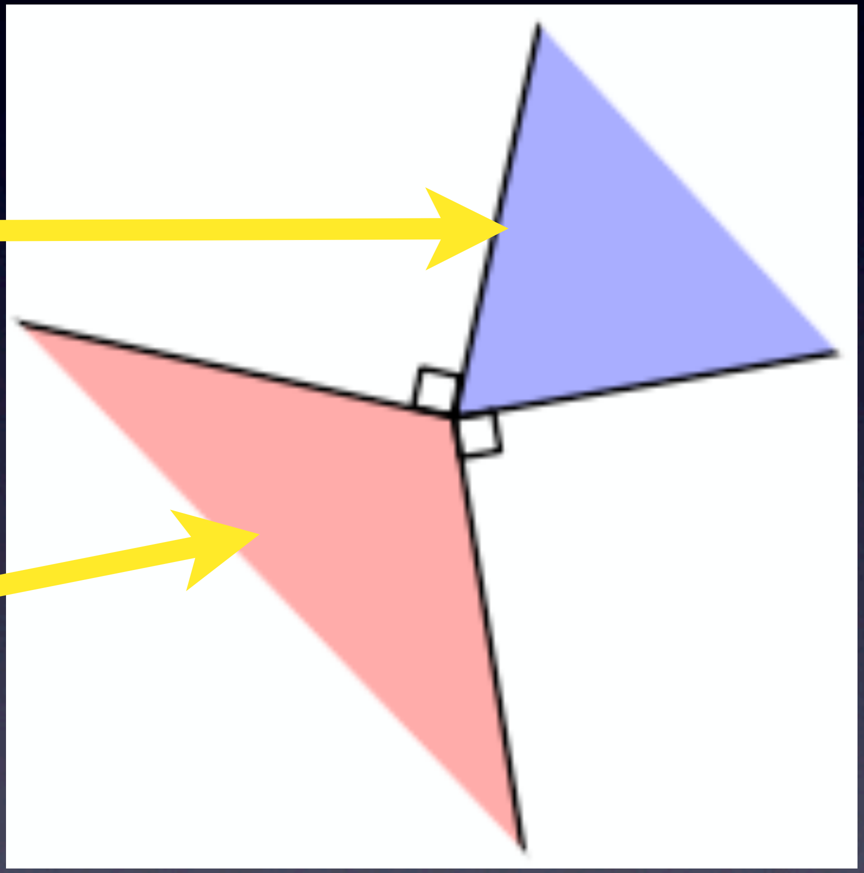

Can be converted
into approachability
for the set K^0

$$\frac{1}{T}\text{Regret}_T(K) \leq O(\text{dist}(\tfrac{1}{T}\ell_t, K^0))$$

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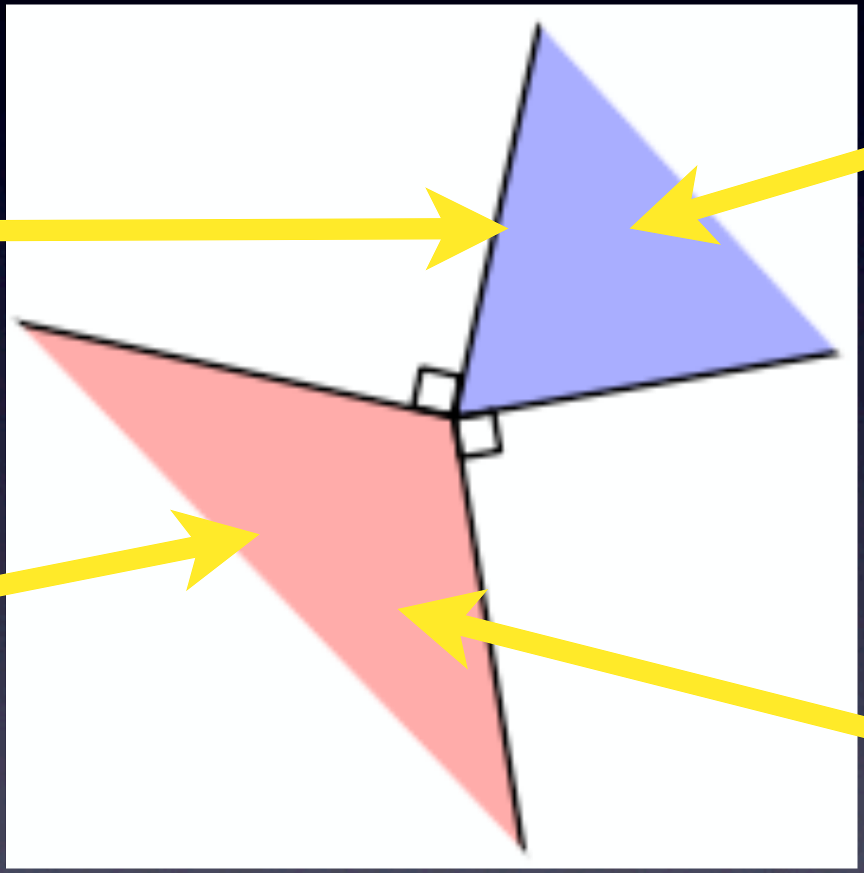
Learning/ optimization in the set K

Approachability of the set S



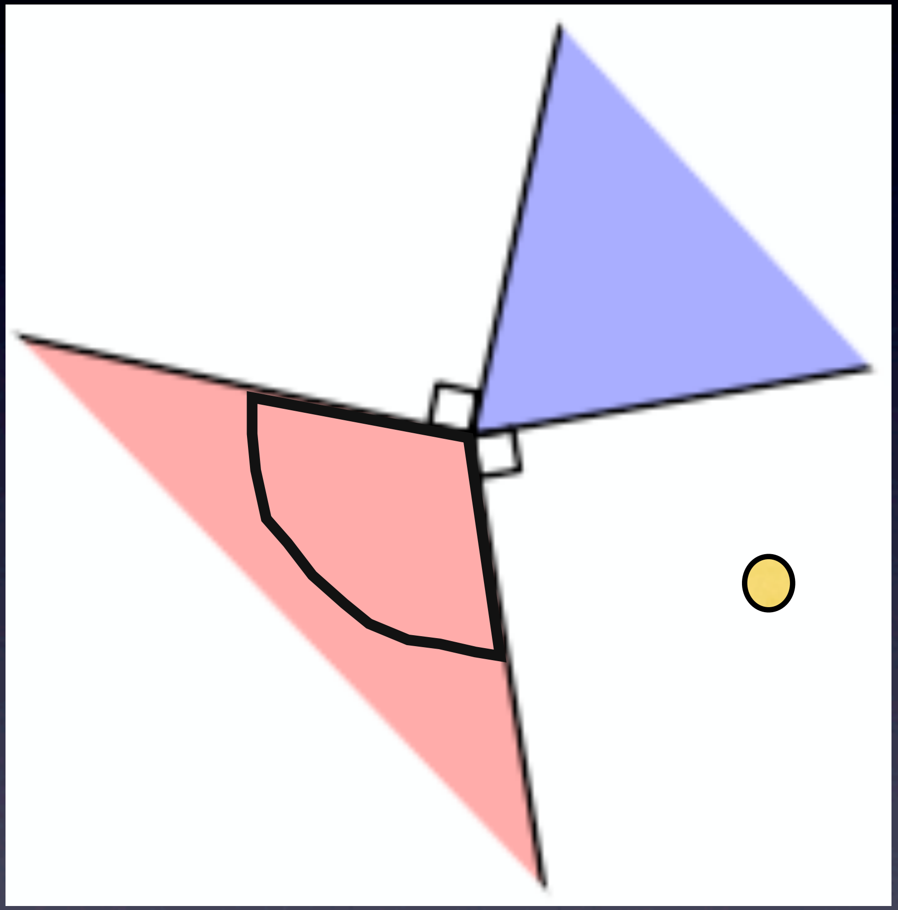
Can be converted into approachability for the set K^0

Is equivalent to regret-min. in the set S^0

$$\frac{1}{T}\text{Regret}_T(K) \leq O(\text{dist}(\tfrac{1}{T}\ell_t, K^0))$$

$$\text{dist}(\tfrac{1}{T}\ell_t, S) \leq O\left(\tfrac{1}{T}\text{Regret}_T(S^0 \cap B_2)\right)$$

Key Lemma



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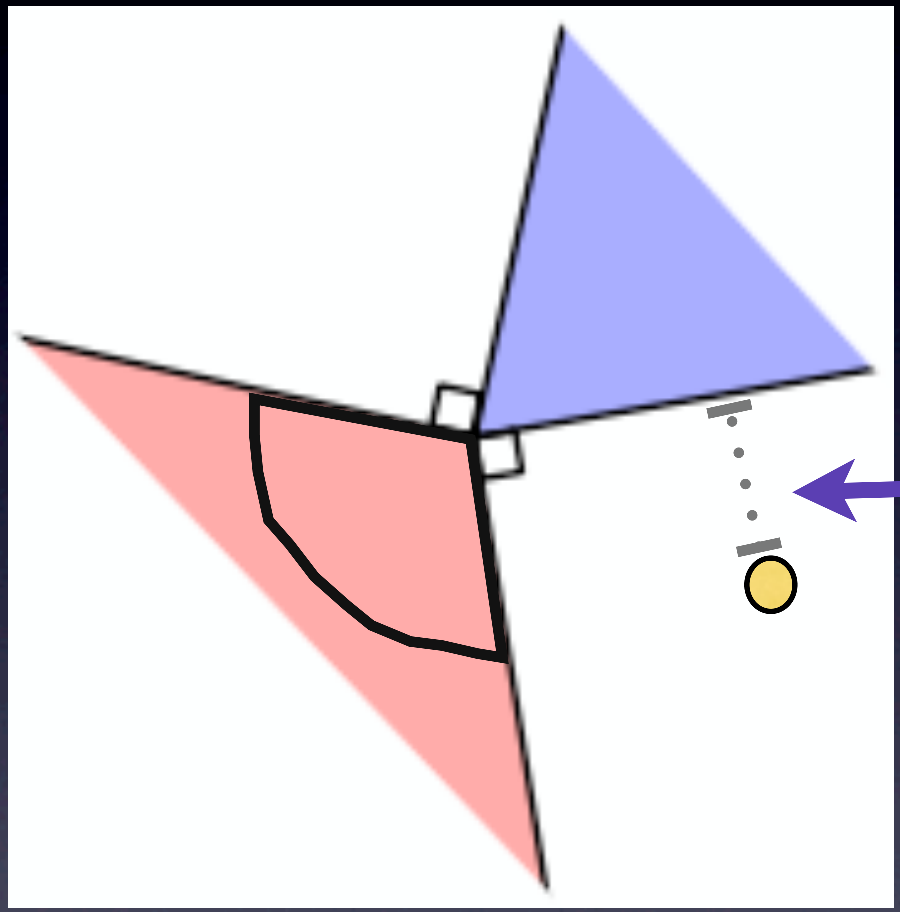

dist(x,cone)

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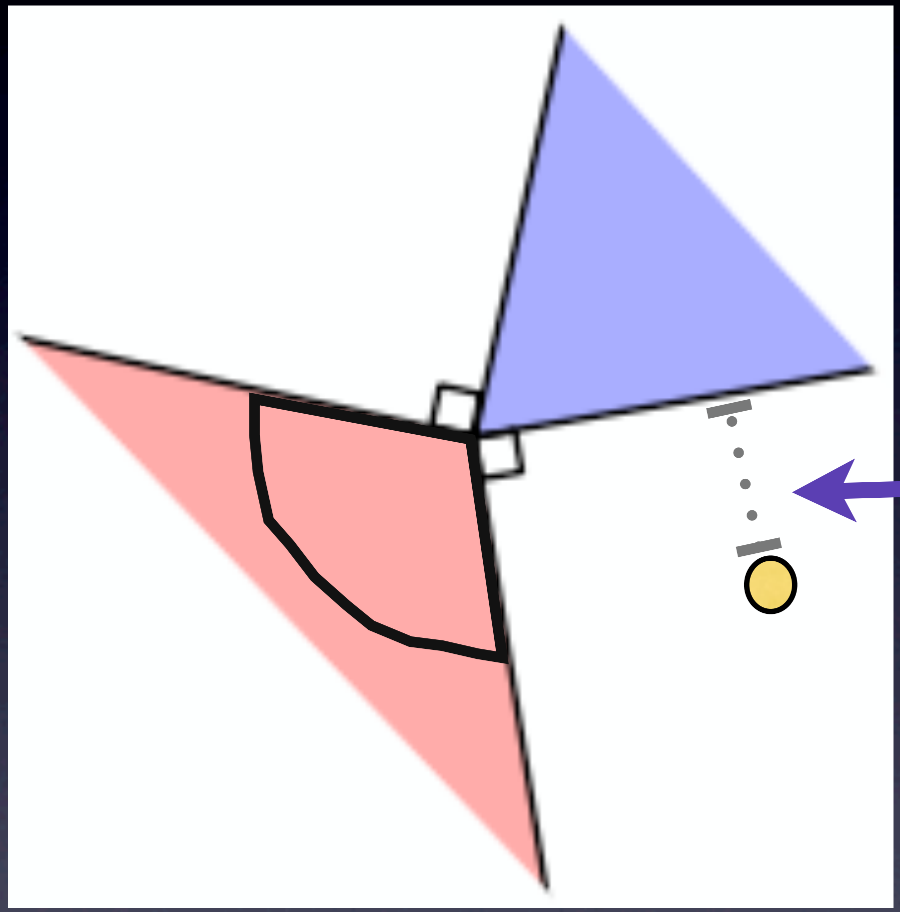

dist(x,cone)

$$\text{dist}(x, \text{cone}) = \max_{v \in \text{cone}^0 \cap \text{ball}(1)} v \cdot x$$

Applications + Conclusions

- Generalization of Blackwell approachability to function spaces
- Efficient binary calibrated forecasting
- New characterization of approachability

Calibrated Forecasting

- Input: a 0/1 bit sequence z_1, z_2, z_3... (i.e. rain, shine, shine, rain)
- Want: "good" probability predictions p1, p2, p3, ...
- "Good" means "for all the times I said 40%, it rained roughly 2/5 of the time", and the same for 10%, 20%, etc.

First Efficient Alg for Binary Calibration

Calibration ==> Approachability ==> Regret-min.

First Efficient Alg for Binary Calibration

Discovered by Foster ’99 ↓

New Approachability to OLO reduction ↓

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↑ We show how to do this in O(log 1/ε) via binary search (requires O(1/ε) memory)

↑ We show how to do this in O(1) via "smart" learning algorithm

Thanks!