Closeness and Distance Relations in Order of Magnitude Qualitative Reasoning via PDL

A. Burrieza E. Muñoz-Velasco M. Ojeda-Aciego

University of Málaga (Spain)

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Outline

Using Logic in Qualitative Reasoning

Different approaches have been proposed which use logic in QR and study the soundness of the reasoning supported by the formalism, together with the efficiency of its use.

- Region Connection Calculus for managing qualitative spatial reasoning [Bennett, Randel et al.]
- Multimodal logics to deal with qualitative spatio-temporal representations [Bennett et al., Wolter et al.]
- Branching temporal logics to describe the possible solutions of ordinary differential equations [Shults et al.].
- Multimodal logics dealing with OMR [Burrieza, Muñoz and Ojeda] on the basis of qualitative classes obtained from the real line divided in intervals.

Order of Magnitude Reasoning

A form of qualitative reasoning is to manage numerical data in terms of orders of magnitude.
[Raiman, Dague, Travé-Massuyès, ...].

- Absolute Order of Magnitude (AOM), which is represented by a partition of the real line $\mathbb{R}$ and each element of $\mathbb{R}$ belongs to a qualitative class.
- Relative Order of Magnitude (ROM), introducing a family of binary order of magnitude relations which establish different comparison relations in $\mathbb{R}$ (e.g., *comparability*, *negligibility* and *closeness*).

Introducing Propositional Dynamic Logic

. . . and showing its advantages

- We present a Propositional Dynamic Logic with constants representing different qualitative classes
  [Harel, Blackburn Van Benthem, Areces TenCate]
- We exploit the possibility of constructing complex relations from simpler ones for defining the concepts of closeness and distance, together with other programming commands such as *while* . . . *do* and *repeat* . . . *until*
- We employ its theoretical support in order to study the decidability of the satisfiability problem.
- Recent applications of PDL in communication scenarios, message-passing systems, multiagent real-time systems and knowledge acquisition.
  [Benthem, Heinemann, Bugaychenko, Bollig]

Outline

Some formulas for qualitative arithmetic

- $\langle +_{\mathsf{ps}} \rangle \varphi$ is true in u iff there exists u', obtained by adding a positive small number to u, such that φ is true in u'.
- $\langle \mathsf{nl?} \rangle \varphi$ is true in u iff u is a negative large number and φ is true in u.
- $\langle +^*_{\mathsf{ps}} \rangle \varphi$ is true in u iff there exists u', obtained by adding a finitely many small positive numbers to u, such that φ is true in u'.

Closeness and Distance

$$[c]\,\varphi = [+_{ns} \cup +_0 \cup +_{ps}]\varphi$$

$[c]\,\varphi$ is true in u iff for every u' which is *close to* u, i.e obtained by adding a small number to u, φ is true in u'.

$$[d]\,\varphi = [+_{nl} \cup +_{pl}]\varphi$$

$[d]\,\varphi$ is true in u iff for every u' which is *distant from* u, i.e obtained by adding a large number to u, φ is true in u'.

A device to control the temperature

Assumptions

- The qualitative classes $\mathrm{NL}, \mathrm{NM}, \mathrm{NS} \cup \mathrm{PS}, \mathrm{PM}$ and PL are interpreted by the formulas: VERY_COLD, COLD, OK, HOT and VERY_HOT, respectively.
- Program $+_0$ means that the system is *off*; moreover $+_{\mathsf{ps}} \cup +_{\mathsf{pm}}$ and $+_{\mathsf{pl}}$, mean that a system for *heating* and *extra heating* are operating, respectively.
- Similarly we consider the meanings of programs $+_{\mathsf{nm}} \cup +_{\mathsf{ns}}$ and $+_{\mathsf{nl}}$ for *cooling* and *extra cooling* operations, respectively.

A device to control the temperature

Some valid formulas

1. $\textsc{Hot} \longrightarrow ([+_{\mathsf{pl}}]\textsc{Very\_Hot} \wedge \langle(+_{\mathsf{nm}} \cup +_{\mathsf{ns}})^*\rangle\textsc{Ok})$
2. *while. . . do*
   $[(\neg\textsc{Ok}?; +_{\mathsf{Sys}})^*; \textsc{Ok}?]\textsc{Ok}$
   $+_{\mathsf{Sys}} = +_{\mathsf{nl}} \cup +_{\mathsf{nm}} \cup +_{\mathsf{ns}} \cup +_{\mathsf{ps}} \cup +_{\mathsf{pm}} \cup +_{\mathsf{pl}}$
3. *repeat. . . until*
   $\textsc{Very\_Hot} \longrightarrow [(+_{\mathsf{nl}}; (\neg\textsc{Ok}?; +_{\mathsf{nl}})^*; \textsc{Ok}?]\textsc{Ok}$
4. $0 \longrightarrow [\mathsf{c}]\,\textsc{Ok}$
5. $\textsc{Ok} \longrightarrow [\mathsf{d}]\,(\textsc{Very\_Cold} \vee \textsc{Cold} \vee \textsc{Hot} \vee \textsc{Very\_Hot})$
6. $\textsc{Ok} \longrightarrow ([\mathsf{c}]\,\textit{efficient} \wedge [\mathsf{d}]\,\textit{warning})$

Outline

Axiom schemata for qualitative classes

QE $\mathsf{nl} \vee \mathsf{nm} \vee \mathsf{ns} \vee 0 \vee \mathsf{ps} \vee \mathsf{pm} \vee \mathsf{pl}$

QU $\star \longrightarrow \neg\#$ for every $\star \in \mathbb{C}$ and $\# \in \mathbb{C} - \{\star\}$

QO1 $\mathsf{nl} \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, \mathsf{nm}$

QO2 $\mathsf{nm} \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, \mathsf{ns}$

QO3 $\mathsf{ns} \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, 0$

QO4 $0 \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, \mathsf{ps}$

QO5 $\mathsf{ps} \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, \mathsf{pm}$

QO6 $\mathsf{pm} \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, \mathsf{pl}$

Axiom schemata for specific programs

PS1 $\mathsf{nl} \longrightarrow [+_{\mathsf{ps}}]\,(\mathsf{nl} \vee \mathsf{nm})$

PS2 $\mathsf{nm} \longrightarrow [+_{\mathsf{ps}}]\,(\mathsf{nm} \vee \mathsf{ns})$

PS3 $\mathsf{ns} \longrightarrow [+_{\mathsf{ps}}]\,(\mathsf{ns} \vee 0 \vee \mathsf{ps})$

PS4 $\mathsf{ps} \longrightarrow [+_{\mathsf{ps}}]\,(\mathsf{ps} \vee \mathsf{pm})$

PS5 $\mathsf{pm} \longrightarrow [+_{\mathsf{ps}}]\,(\mathsf{pm} \vee \mathsf{pl})$

PS6 $\mathsf{pl} \longrightarrow [+_{\mathsf{ps}}]\,\mathsf{pl}$

Z1 $\langle +_0 \rangle\, \varphi \longrightarrow [+_0]\, \varphi$

Z2 $[+_0]\, \varphi \longrightarrow \varphi$

Decidability results

Lemma (Filtration Lemma)

Let (W, m) be a model and $(\overline{W}, \overline{m})$ defined as previously from a formula φ. Consider $u, v \in W$.

1. *For all $\psi \in FL(\varphi)$, $u \in m(\psi)$ iff $\overline{u} \in \overline{m}(\psi)$.*
2. *For all $[a]\psi \in FL(\varphi)$,*
   - *if $(u, v) \in m(a)$ then $(\overline{u}, \overline{v}) \in \overline{m}(a)$;*
   - *if $(\overline{u}, \overline{v}) \in \overline{m}(a)$ and $u \in m([a]\psi)$, then $v \in m(\psi)$.*

Theorem (Small Model Theorem)

Let φ a satisfiable formula, then φ is satisfied in a model with no more than $2^{|\varphi|}$ states.

Outline

Conclusions

- A PDL for order of magnitude reasoning has been introduced which deals with qualitative relations such as closeness and distance.
- An axiom system for this logic has been defined by including as axioms the formulas which express syntactically the required properties.
- We have shown the decidability of the satisfiability problem of our logic.

Future work

- We are currently studying the completeness of the proposed axiom system, as well as the complexity of the satisfiability problem.
- We are trying to extend this approach for more relations such as a linearity and negligibility, by maintaining decidability and completeness.
- We have planned to give a relational proof system based on dual tableaux for this logic.
- We are looking for more applications of our approach.

Thank you and Contact

THANK YOU VERY MUCH!!!

CONTACT:
Emilio Muñoz-Velasco
Dept. Applied Mathematics
University of Málaga
Spain
emilio@ctima.uma.es
http://www.matap.uma.es/~emilio/

Outline

The Language

Formulas:

- $\Phi_0 = \mathbb{V} \cup \mathbb{C}$, where $\mathbb{V}$ is a denumerable set of propositional variables and $\mathbb{C} = \{\mathrm{nl}, \mathrm{nm}, \mathrm{ns}, 0, \mathrm{ps}, \mathrm{pm}, \mathrm{pl}\}$.
- If φ and ψ are formulas and *a* is a program, then $\varphi \longrightarrow \psi$ (propositional implication), $\perp$ (propositional falsity) and $[a]\varphi$ (program neccesity) are also formulas.

Programs:

- $\Pi_0 = \{+_\star \mid \star \in \mathbb{C}\}$.
- If *a* and *b* are programs and φ is a formula, then $(a; b)$ ("do *a* followed by *b*"), $a \cup b$ ("do either *a* or *b*, nondeterministically"), a^* ("repeat *a* a nondeterministically chosen finite number of times") and $\varphi?$ ("proceed if φ is true, else fail") are also programs.

The Semantics

We now define the *semantics* of our logic. A *model* $\mathcal{M}$ is a tuple (W, m), where W is a non-empty set divided in 7 qualitative classes, $\{\mathrm{nl}, \mathrm{nm}, \mathrm{ns}, 0, \mathrm{ps}, \mathrm{pm}, \mathrm{pl}\}$ and m is a meaning function such that $m(p) \subseteq W$, for every propositional variable, $m(\star) = \star$, for every $\star \in \mathbb{C}$ and $m(a) \subseteq W \times W$, for all program a. Moreover, for every formula φ and ψ and for all programs a, b, we have:

- $m(\bot) = \varnothing \quad m(\varphi \longrightarrow \psi) = (W \smallsetminus m(\varphi)) \cup m(\psi)$
- $m([a]\varphi) = \{w \in W : \text{for all } v \in W, \text{ if } (w, v) \in m(a) \text{ then } v \in m(\varphi)\}$
- $m(a \cup b) = m(a) \cup m(b)$
- $m(a; b) = m(a); m(b)$
- $m(a^*) = m(a)^*$
- $m(\varphi?) = \{(w, w) : w \in m(\varphi)\}$

The properties of qualitative sum

1. $m(+_{\text{ps}})(\text{nl}) \subseteq \text{nl} \cup \text{nm}$
2. $m(+_{\text{ps}})(\text{nm}) \subseteq \text{nm} \cup \text{ns}$
3. $m(+_{\text{ps}})(\text{ns}) \subseteq \text{ns} \cup 0 \cup \text{ps}$
4. $m(+_{\text{ps}})(\text{ps}) \subseteq \text{ps} \cup \text{pm}$
5. $m(+_{\text{ps}})(\text{pm}) \subseteq \text{pm} \cup \text{pl}$
6. $m(+_{\text{ps}})(\text{pl}) \subseteq \text{pl}$

The rest of axioms for qualitative sum

Positive medium numbers

PM1 $\mathsf{nl} \longrightarrow [+_{\mathsf{pm}}]\,(\mathsf{ns} \vee \mathsf{nm} \vee \mathsf{nl})$

PM2 $\mathsf{nm} \longrightarrow [+_{\mathsf{pm}}]\,(\mathsf{nm} \vee \mathsf{ns} \vee 0 \vee \mathsf{ps} \vee \mathsf{pm})$

PM3 $\mathsf{ns} \longrightarrow [+_{\mathsf{pm}}]\,(\mathsf{ps} \vee \mathsf{pm})$

PM4 $\mathsf{ps} \longrightarrow [+_{\mathsf{pm}}]\,(\mathsf{pm} \vee \mathsf{pl})$

PM5 $\mathsf{pm} \longrightarrow [+_{\mathsf{pm}}]\,(\mathsf{pm} \vee \mathsf{pl})$

PM6 $\mathsf{pl} \longrightarrow [+_{\mathsf{pm}}]\,\mathsf{pl}$

The rest of axioms for qualitative sum

Positive large numbers

PL1 $\mathsf{nm} \longrightarrow [+_{\mathsf{pl}}]\,(\mathsf{ps} \vee \mathsf{pm} \vee \mathsf{pl})$

PL2 $\mathsf{ns} \longrightarrow [+_{\mathsf{pl}}]\,(\mathsf{pm} \vee \mathsf{pl})$

PL3 $\mathsf{ps} \longrightarrow [+_{\mathsf{pl}}]\,\mathsf{pl}$

PL4 $\mathsf{pm} \longrightarrow [+_{\mathsf{pl}}]\,\mathsf{pl}$

PL5 $\mathsf{pl} \longrightarrow [+_{\mathsf{pl}}]\,\mathsf{pl}$

Fisher-Ladner Closure

$$FL : \Phi \longrightarrow 2^{\Phi} \quad FL^{\Box} : \{[a]\varphi \mid a \in \Pi, \varphi \in \Phi\} \longrightarrow 2^{\Phi}$$

(a) $FL(p) = \{p\}$, for every propositional variable p.

(b) $FL(\star) = \star$, for all $\star \in \mathbb{C}$.

(c) $FL(\varphi \longrightarrow \psi) = \{\varphi \longrightarrow \psi\} \cup FL(\varphi) \cup FL(\psi)$

(d) $FL(\bot) = \{\bot\}$

(e) $FL([a]\varphi) = FL^{\Box}([a]\varphi) \cup FL(\varphi)$

(f) $FL^{\Box}([a]\varphi) = \{[a]\varphi\}$, being a an atomic program.

(g) $FL^{\Box}([a \cup b]\varphi) = \{[a \cup b]\varphi\} \cup FL^{\Box}([a]\varphi) \cup FL^{\Box}([b]\varphi)$

(h) $FL^{\Box}([a; b]\varphi) = \{[a; b]\varphi\} \cup FL^{\Box}([a][b]\varphi) \cup FL^{\Box}([b]\varphi)$

(i) $FL^{\Box}([a^*]\varphi) = \{[a^*]\varphi\} \cup FL^{\Box}([a][a^*]\varphi)$

(j) $FL^{\Box}([\psi?]\varphi) = \{[\psi?]\varphi\} \cup FL(\psi)$

The filtration structure

The filtration structure $(\overline{W}, \overline{m})$ of (W, m) by $FL(\varphi)$ is defined on the quotient set $W/\equiv$, denoted by $\overline{W}$, and the qualitative classes in $\overline{W}$ are defined, for every $\star \in \mathbb{C}$, by $\overline{\star} = \{\overline{u} \mid u \in \star\}$. Furthermore, the map $\overline{m}$ is defined as follows:

1. $\overline{m}(p) = \{\overline{u} \mid u \in m(p)\}$, for every propositional, variable p.
2. $\overline{m}(\star) = m(\star) = \star$, for all $\star \in \mathbb{C}$.
3. $\overline{m}(a) = \{(\overline{u}, \overline{v}) \mid \exists u' \in \overline{u}$ and $\exists v' \in \overline{v}$ such that $(u', v') \in m(a)\}$, for every atomic program a.

$\overline{m}$ is extended inductively to compound propositions and programs as described previously in the definition of model.

Outline

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