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Approximation of Random Fields in High Dimension

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Definition of tensor product-type random fields

- for any $d = 1, 2, \ldots$: separable zero-mean random function $\{X^{(d)}(t)\}$, $t \in [0,1]^d$ with the covariance function

$$\mathcal{K}^{(d)}(s,t) = \prod_{\ell=1}^{d} \mathcal{K}_\ell(s_\ell, t_\ell)$$

for all $s_\ell, t_\ell \in [0,1]$, $s = (s_1, \ldots, s_d)$, $t = (t_1, \ldots, t_d)$.

- family of tensor product-type random fields

$$\mathbb{X} = \left\{ X^{(d)}(t), t \in [0,1]^d \right\}, \quad d = 1, 2, \ldots .$$

$d \to \infty$.

Some examples of tensor product-type random fields

- Wiener-Chentsov random field:

$$\mathcal{K}_{W^{(d)}}(s,t) = \prod_{\ell=1}^{d} \min\{s_\ell, t_\ell\}.$$

- d-variate completely tucked Brownian sheet (Hoefding, Blum, Kiefer and Rosenblatt process):

$$\mathcal{K}_{B^{(d)}}(s,t) = \prod_{l=1}^{d} \left(\min\{s_\ell, t_\ell\} - s_\ell t_\ell\right).$$

$s_\ell, t_\ell \in [0,1]$, $s = (s_1, \ldots, s_d)$, $t = (t_1, \ldots, t_d)$.

Karhunen-Loève expansion

$$\Lambda := \sum_{i=1}^{\infty} \lambda_i^2 < \infty.$$

λ_i^2 and $\varphi_i(\cdot)$: $\lambda_i^2 \varphi_i(t) = \int_0^1 \mathcal{K}(s,t)\varphi_i(s)\mathrm{d}s, \; t \in [0,1]$.

The covariance operator of $X^{(d)}$ has the eigenvalues:

$$\lambda_{\boldsymbol{k}}^2 := \lambda_{k_1}^2 \lambda_{k_2}^2 \dots \lambda_{k_d}^2, \; \boldsymbol{k} = (k_1, k_2, \dots, k_d) \in \mathbb{N}^d.$$

Karhunen-Loève expansion: with $\boldsymbol{\varphi}_{\boldsymbol{k}}(\boldsymbol{t}) = \varphi_{k_1}(t_1) \dots \varphi_{k_d}(t_d)$

$$X^{(d)}(\boldsymbol{t}) = \sum_{\boldsymbol{k} \in \mathbb{N}^d} \xi_{\boldsymbol{k}} \lambda_{\boldsymbol{k}} \boldsymbol{\varphi}_{\boldsymbol{k}}(\boldsymbol{t}), \; \boldsymbol{t} = (t_1, \dots, t_d) \in [0,1]^d, \qquad (1.1)$$

where $\{\xi_{\boldsymbol{k}}\}$ is an array of non-correlated $\mathcal{N}(0,1)$ r.v.'s.

Average case information complexity

Let $X := X^{(d)}$ and X_n be the partial sum of the Karhunen-Loève expansion (1.1) corresponding to n maximal eigenvalues.

The average case information complexity:

$$n(\varepsilon, d) := \min\{n : \mathbb{E}\|X - X_n\|^2 \le \varepsilon^2 \Lambda^d\},$$

where

$$\Lambda^d = \left(\sum_{i=1}^{\infty} \lambda_i^2\right)^d = \sum_{\boldsymbol{k} \in \mathbb{N}^d} \lambda_{\boldsymbol{k}}^2 = \mathbb{E}\|X\|^2.$$

Aim: the asymptotic behavior of $n(\varepsilon, d)$, as $d \to \infty$.

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Auxiliary sequence of i.i.d. r.v.'s

i.i.d. $\{U_\ell\}$: $\mathbb{P}(U_\ell = -\log\lambda_i) = \lambda_i^2/\Lambda,\ i = 1, 2, \dots$.

(3dM): $\sum_{i=1}^{\infty} |\log\lambda_i|^3 \lambda_i^2 < \infty$, then $\mathbb{E}|U_\ell|^3 < \infty$.

$$M := \mathbb{E}U_\ell = -\sum_{i=1}^{\infty} \log\lambda_i \frac{\lambda_i^2}{\Lambda},$$

$$\sigma^2 := \mathrm{Var}\, U_\ell = \sum_{i=1}^{\infty} |\log\lambda_i|^2 \frac{\lambda_i^2}{\Lambda} - M^2.$$

$$\alpha^3 := \mathbb{E}(U_\ell - M)^3 = -\sum_{i=1}^{\infty} (\log\lambda_i)^3 \frac{\lambda_i^2}{\Lambda} - 3M\sigma^2 - M^3.$$

Explosion coefficient

$\mathcal{E} := \Lambda e^{2M}$.

Lemma

$$\begin{aligned} \mathcal{E} &> 1. \\ \mathcal{E} &= 1 \iff \sigma = 0. \end{aligned}$$

Proof: by concavity of the logarithmic function.

Information complexity

Recall: $\mathcal{E} = \Lambda e^{2M} > 1$.

Theorem

Let $\{\lambda_i\}_{i\geq 1}$ satisfy (3dM). Then for every $\varepsilon \in (0,1)$ it holds

$$n(\varepsilon, d) = K\ \phi(q_\varepsilon/\sigma)\ \mathcal{E}^d e^{2q_\varepsilon\sqrt{d}}\, d^{-1/2}\,(1+o(1))\,, \quad d \to \infty,$$

where $\phi(\cdot)$ is the standard normal density, the constant K is known and depends on whether U_ℓ has a lattice distribution or not, and the quantile $q = q_\varepsilon$ is chosen from the equation

$$1 - \Phi\,(q/\sigma) = \varepsilon^2.$$

Curse of dimensionality: $\log n(\varepsilon, d) = d \log \mathcal{E}\,(1+o(1)),\ d \to \infty.$

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First step: sum of units

$\zeta := \zeta(\varepsilon, d)$: the maximal positive number s.t.:

$$\sum_{\boldsymbol{k}\in\mathbb{N}^d:\lambda_{\boldsymbol{k}}<\zeta} \lambda_{\boldsymbol{k}}^2 = \mathbb{E}\|X - X_n\|^2 \leq \varepsilon^2 \Lambda^d.$$

$$\mathrm{A} = \mathrm{A}(\varepsilon, d) := \left\{\boldsymbol{k} \in \mathbb{N}^d : \lambda_{\boldsymbol{k}} \geq \zeta\right\} = \left\{\boldsymbol{k} \in \mathbb{N}^d : \prod_{\ell=1}^{d} \lambda_{k_\ell} \geq \zeta\right\}.$$

First step: sum of units. (Cont.)

Remember: $\lambda_i^2 = \Lambda\,\mathbb{P}(U_\ell = -\log\lambda_i),\ i = 1, 2, \dots.$

$\lambda_{\boldsymbol{k}} > 0$ for any $\boldsymbol{k} \in \mathrm{A}$:

$$n(\varepsilon, d) = card\,(\mathrm{A}) = \sum_{\boldsymbol{k}\in\mathrm{A}} \frac{\lambda_{\boldsymbol{k}}^2}{\lambda_{\boldsymbol{k}}^2}$$

$$= \sum_{\boldsymbol{k}\in\mathbb{N}^d:\sum U_\ell\le -\log\zeta} \exp\Big\{-2\sum_{\ell=1}^{d}\log\lambda_{k_\ell}\Big\}\Lambda^d\prod_{\ell=1}^{d}\mathbb{P}(U_\ell = -\log\lambda_{k_\ell})$$

$$= \Lambda^d\,\mathbb{E}\exp\Big\{2\sum_{\ell=1}^{d}U_\ell\Big\}\mathbb{I}_{\{\sum_{\ell=1}^{d}U_\ell\le -\log\zeta\}}.$$

Second step: quantile convergence

For any $d \in \mathbb{N}$, $z \in \mathbb{R}^1$ and fixed $\varepsilon \in (0,1)$:

$$\sum_{\mathbf{k}\in\mathbb{N}^d:\lambda_{\mathbf{k}}<z} \lambda_{\mathbf{k}}^2 = \Lambda^d \, \mathbb{P}\left(\sum_{l=1}^{d} U_l > -\log z\right)$$

$$= \Lambda^d \, \mathbb{P}\left(Z_d > -\frac{\log z + dM}{\sigma\sqrt{d}}\right) = \Lambda^d \, \mathbb{P}\left(Z_d > \theta_z\right) \leq \varepsilon^2 \Lambda^d,$$

where $\theta_z = -\frac{\log z + dM}{\sigma\sqrt{d}}$, and $Z_d := \frac{\sum_{\ell=1}^{d} U_\ell - dM}{\sigma\sqrt{d}}$.

Then $\theta = \theta(\varepsilon, d)$ is the $(1-\varepsilon^2)$-quantile of the d.f. of Z_d.
By the CLT: $\theta(\varepsilon, d) \to q_\varepsilon/\sigma$, $d \to \infty$, where $q = q_\varepsilon$ is the quantile of the normal d.f.

Thus, for the information complexity we obtained:

$$
\begin{aligned}
n(\varepsilon, d) &= \mathcal{E}^d \, \mathbb{E} \exp\{2\sigma\sqrt{d} Z_d\} \mathbb{I}_{\{Z_d \le \theta\}} \\
&= \mathcal{E}^d \exp\{2\sigma\sqrt{d}\theta\} \int_{-\infty}^{\theta} \exp\{2\sigma\sqrt{d}(z-\theta)\} \, \mathrm{d}F_d(z),
\end{aligned}
$$

where $F_d(z) = \mathbb{P}(Z_d < z)$, $\mathcal{E} := \Lambda e^{2M}$,

$$
Z_d := \frac{\sum_{\ell=1}^{d} U_\ell - dM}{\sigma\sqrt{d}},
$$

$$
\theta = \theta(\varepsilon, d) := -\frac{\log\zeta + dM}{\sigma\sqrt{d}}.
$$

Third step: application of the Edgeworth-type expansions

- The rest of the proof:
  - integrate by parts
  - apply the Cramér-Esseen Theorems:

if the distribution of U_ℓ is not lattice, it holds (uniformly in z):

$$F_d(z) = \Phi(z) - \frac{e^{-z^2/2}}{\sqrt{2\pi}} \frac{\alpha^3(z^2-1)}{6\sigma^3\sqrt{d}} + o\left(\frac{1}{\sqrt{d}}\right)$$

and with the additional periodic term otherwise:

$$\frac{e^{-z^2/2} hS\left((z\sigma\sqrt{d} - da)/h\right)}{\sigma\sqrt{2\pi d}},$$

where $S(x) := [x] - x + \frac{1}{2}$, a is a shift and h is the maximal span of the distribution of $U_\ell - M$.

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References

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