18.03 Problem Set 5 Solutions: Part II

Each problem is worth 16 points, spread across Parts I and II. Part I values: **20** 3 points; **21** 2 points.

17. (c) [6] We have to check that $|(1/2) - (1/(1+bi)| = 1/2$. There are many ways to do this. Here is one:

$$\left|\frac{1}{2} - \frac{1}{1+bi}\right| = \left|\frac{1+bi-2}{2(1+bi)}\right| = \frac{1}{2}\cdot\frac{|bi-1|}{|bi+1|}.$$

But $+1$ and -1 are equidistant from bi, so the second factor is 1.

(b) [5] The clever way to do this is to think of the trajectory of $W(i\omega)$, that, is, the curve that it parametrizes in the complex plane. It's a circle of radius 1/2 and center 1/2. The gain is the distance from the origin. This equals $1/\sqrt{2}$ when the angle is $\pm\pi/4$. One way to see this is to write the point on the circle as $1/(1+bi)$, as in **(c)**, and observe that $|1/(1+bi)| = \sqrt{2}$ just when $b = \pm 1$. So $-\phi = \pm\pi/4$.

(a) [5] Now we know that gain of $1/\sqrt{2}$ occurs when $W(i\omega) = (1 \pm i)/2$. The handout has a nice expression for $W(i\omega)^{-1}$ which we will use. From the fact that points on the circle are the reciprocals of points on the line with real part 1, or by direct calculation, we find $((1\mp i)/2)^{-1} = 1 \pm i$, so $1 - \dfrac{i}{b/m}\dfrac{\omega_n^2 - \omega^2}{\omega} = 1 \pm i$. Thus $\omega_n^2 - \omega^2 = \pm b\omega/m$. These quadratic equations have solutions $\omega = \pm(b/2m) \pm \sqrt{(b/2m)^2 + \omega_n^2}$, where the signs are independent. The second term is larger in absolute value than the first, so the positive solutions are the square root plus or minus the first term, and differ by twice $b/2m$, or b/m.

18. (a) [4] We have found that homogeneous linear equations have solutions with more than one extreme point only in the underdamped case. In that case, we know that successive extrema of solutions are separated by half the period, so, from what we've been told, $\pi/\omega_d = \pi$ or $\omega_d = 1$. The solution has the form $x = Ae^{-bt/2}\cos(\omega_d t - \phi)$, and when t is increased by half a period the cosine simply changes sign. Since the half-period is π, $x(\pi) = -1/2$ implies that $1/2 = -x(\pi)/x(0) = Ae^{-b\pi/2}/A = e^{-b\pi/2}$. Thus $b = 2(\ln 2)/\pi$. But $1 = \omega_d^2 = \omega_n^2 - (b/2)^2$, so $k = \omega_n^2 = 1 + ((\ln 2)/\pi)^2$.

(b) [4] Just substitute this in:

$$\begin{array}{|llcr|} \hline k] & x & = & 2\sin(2t) \\ b] & \dot{x} & = & 4\cos(2t) \\ 1] & \ddot{x} & = & -8\sin(2t) \\ \hline & \cos(2t) & = & (2k-8)\sin(2t) + 4b\cos(2t) \\ \hline \end{array}$$

so (since $\cos(2t)$ and $\sin(2t)$ are linearly independent) $k = 4$ and $b = 1/4$.

(c) [4] Just substitute this in:

$$\begin{array}{|llcr|} \hline k] & x & = & 1 + e^{-t}\sin t \\ b] & \dot{x} & = & e^{-t}(-\sin t + \cos t) \\ 1] & \ddot{x} & = & -2e^{-t}\cos t \\ \hline & 2 & = & k + e^{-t}((k-b)\sin t + (b-2)\cos t) \\ \hline \end{array}$$

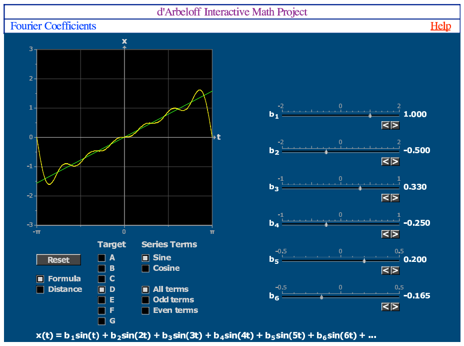
so (since 1, $e^{-t}\cos t$ and $e^{-t}\sin t$ are linearly independent) $k = 2$ and $b = 2$. [Notice that this would be forced even if you only knew that the input was constant.]

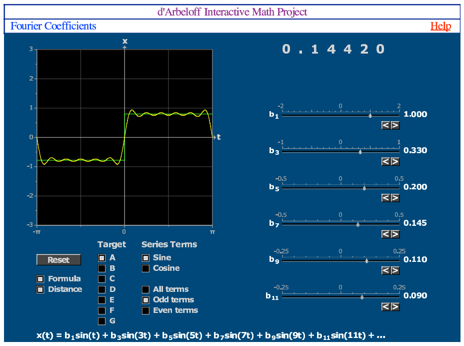
(d) [4] This is the imaginary part of the complex equation is $\ddot{z} + z = te^{it}$. Look for a solution of the form $z = e^{it}u$. If we substitute this in, $\dot{z} = e^{it}(\dot{u} + iu)$, $\ddot{z} = e^{it}(\ddot{u} + 2i\dot{u} - u)$, so $e^{it}t = \ddot{z} + z = e^{it}(\ddot{u} + 2i\dot{u})$. Cancel the exponental: $\ddot{u} + 2i\dot{u} = t$. (We could also have used ESL: $p(s) = s^2 + 1$, $p(D)(e^{it}u) = e^{it}p(D + iI)u$, and $p(D + iI) = (D + iI)^2 + I = D^2 + 2iD$, so we arrive at the same result.) Now we have to use reduction of order: $v = \dot{u}$, so $\dot{v} + 2iv = t$.

By undertermined coefficients, try $v_p = at + b$; $\dot{v} = a$, so $t = \dot{v} + 2iv = 2iat + (a + 2ib)$, which implies $a = 1/2i = -i/2$ and then $b = -(1/2i)a = 1/4$, so $v_p = -(i/2)t + (1/4)$. Then so $u_p = -(i/4)t^2 + (1/4)t$, $z_p = (-(i/4)t^2 + (1/4)t)e^{it}$. x_p is the imaginary part of this, which is $x_p = -(t^2/4)\cos t + (t/4)\sin t$.

20. (a) [0] This is very subjective!

(b) [4] $b_n = (2/\pi)\int_0^\pi t/2\sin(nt)\,dt$ which we can integrate by parts: $u = t$, $dv = \sin(nt)\,dt$, $du = dt$, $v = -(1/n)\cos(nt)$, and $\int \cos(nt)\,dt = (1/n)\sin(nt) + c$,
so $b_n = (1/\pi)([-(t/n)\cos(nt)]_0^\pi + (1/n^2)[\sin(nt)]_0^\pi)$. Now $\cos(n\pi) = 1$ for n even and -1 for n odd, and $\sin(n\pi) = 0$ for all n, so $b_n = (1/\pi)(-\pi/n) = -1/n$ for n even and $b_n = -(1/\pi)(-\pi/n) = 1/n$ for n odd: $f(t) = \sin(t) - (1/2)\sin(2t) + (1/3)\sin(3t) - \cdots$. The settings $b_1 = 1.000$, $b_2 = -.500$, $b_3 = .330$, $b_4 = -.250$, $b_5 = .200$, $b_6 = -.165$, lead to a much better approximation!





(c) [3] For n even, the function $\sin(nt)$ is odd about $\pi/2$, while the target function is even about $\pi/2$. This effect may be expressed in many ways. Any initial setting and any sequence of optimizations leads to $b_n = 1/n$ for n odd. These fractions are approximated by $b_1 = 1.000$, $b_3 = .330$, $b_5 = .200$, $b_7 = .140$, $b_9 = .110$ or $.112$, $b_{11} = .090$ or $.092$.

(d) [3] $\sin(t - \pi/4) = -\cos(\pi/4)\cos t + \sin(\pi/4)\sin t = (1/\sqrt{2})(-\cos t + \sin t)$, so $-a_1 = b_1 = 1/\sqrt{2}$ and all the other Fourier coefficients are zero.

(e) [3]

target	sine/cos	even/odd
A	sine	odd
B	cos	odd
C	cos	all
D	sine	odd
E	sine	even
F	cos	even

See the Supplementary Notes §16.4 for more information about this.

21. (a) [3] $|\cos(t/2)|$ is even, so $b_n = 0$ for all n. $\cos(t/2) \geq 0$ for $0 \leq t \leq \pi$, so $a_n = \frac{2}{\pi}\int_0^\pi \cos(t/2)\cos(nt)\,dt$. To integrate this we'll use the trig identity stated in EP 8.1: 27, to see

$$a_n = \frac{1}{\pi}\int_0^\pi (\cos((n+(1/2))t) + \cos((n-(1/2))t))\,dt = \frac{1}{\pi}\left[\frac{\sin((n+(1/2))t)}{n+(1/2)} + \frac{\sin((n-(1/2))t)}{n-(1/2)}\right]_0^\pi .$$

n	$\sin((n+(1/2))\pi)$	$\sin((n-(1/2))\pi)$
0	1	-1
1	-1	1
2	1	-1
$\vdots$	$\vdots$	$\vdots$

so we have to give $\dfrac{1}{n+(1/2)} - \dfrac{1}{n-(1/2)} = -\dfrac{1}{n^2-(1/4)}$ alternating signs:

$$|\cos(t/2)| = \frac{1}{\pi}\left[-\frac{1}{-1/4} + \frac{\cos(t)}{1-(1/4)} - \frac{\cos(2t)}{4-(1/4)} + \cdots\right].$$

(b) [3] With $L = 2\pi$, $b_n = \dfrac{2}{2\pi}\displaystyle\int_0^{2\pi} \mathrm{sq}(t)\sin\left(\frac{n\pi t}{2\pi}\right)dt = \frac{1}{\pi}\int_0^{\pi}\sin\left(\frac{nt}{2}\right)dt - \frac{1}{\pi}\int_{\pi}^{2\pi}\sin\left(\frac{nt}{2}\right)dt$
$= -\dfrac{1}{\pi}\dfrac{\cos(nt/2)}{n/2}\Big|_0^{\pi} + \dfrac{1}{\pi}\dfrac{\cos(nt/2)}{n/2}\Big|_{\pi}^{2\pi} = \dfrac{2}{n\pi}\left[-\left(\cos\left(\frac{n\pi}{2}\right)-1\right) + \left(\cos\left(\frac{2n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)\right)\right]$
$= \dfrac{2}{n\pi}\left[1 - 2\cos\left(\dfrac{\pi n}{2}\right) + \cos\left(\dfrac{2\pi n}{2}\right)\right].$

n	$\cos(n\pi/2)$	$\cos(n\pi)$	$1-2\cos(n\pi/2)+\cos(n\pi)$
0	1	1	0
1	0	-1	0
2	-1	1	4
3	0	-1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

and the pattern repeats. Thus $b_n = (8/n\pi)$ for $n = 2, 6, 10, \ldots$ and zero otherwise, so $\mathrm{sq}(t) = \dfrac{8}{\pi}\left(\dfrac{\sin(2t/2)}{2} + \dfrac{\sin(6t/2)}{6} + \cdots\right) = \dfrac{4}{\pi}\left(\sin(t) + \dfrac{\sin(3t)}{3} + \dfrac{\sin(5t)}{5} + \cdots\right)$. This is *the same* as the Fourier series of $\mathrm{sq}(t)$ when it is regarded as a function of period 2π instead of period 4π. What was b_5 before is now called b_{10}, but in either case it is the coefficient of $\sin(5t)$, and that coefficient is the same in both ways of looking at $\mathrm{sq}(t)$.

(c) [1] $1 + \sin(t) + 2\,\mathrm{sq}(t) = 1 + (1 + (4/\pi))\sin(t) + (4/\pi)((1/3)\sin(3t) + (1/5)\sin(5t) + \cdots)$.

(d) [2] $\mathrm{sq}(t-(\pi/2)) = (4/\pi)(\sin(t-(\pi/2)) + (1/3)\sin(3t-(3\pi/2)) + (1/5)\sin(5t-(5\pi/2)) + \cdots)$. Now $\sin(\theta - (n\pi/2)) = -\cos\theta$ if $n = 1, 5, 9, \ldots$, and $\sin(\theta-(n\pi/2)) = \cos\theta$ if $n = 3, 7, 11, \ldots$, so $\mathrm{sq}(t-(\pi/2)) = (4/\pi)(-\cos(t) + (1/3)\cos(3t) - (1/5)\cos(5t) + \cdots)$.

(e) [3] $g(t)$ satisfes $g'(t) = \mathrm{sq}(t)$ and $g(0) = 0$. The general solution to the ODE is

$$g(t) = \int \mathrm{sq}(t)\,dt = \frac{4}{\pi}\int \sum_{k\text{ odd}} \frac{\sin(kt)}{k}\,dt = -\frac{4}{\pi}\sum_{k\text{ odd}}\frac{\cos(kt)}{k^2} + c.$$

The constant is the average value of $g(t)$, which is $\pi/2$, so $g(t) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{k\text{ odd}}\frac{\cos(kt)}{k^2}$.

(Evaluating this at $t = 0$ gives an identity of Euler's, $\displaystyle\sum_{k\text{ odd}}\frac{1}{k^2} = \frac{\pi^2}{8}$.)

(f) [1] $\mathrm{sq}(\pi t) = (4/\pi)(\sin(\pi t) + (1/3)\sin(3\pi t) + (1/5)\sin(5\pi t) + \cdots)$.

(g) [1] This function can be expressed in terms of the standard squarewave: $h(t) = (1/2)(1 - \mathrm{sq}(2\pi t)) = (1/2) - (2/\pi)(\sin(2\pi t) + (1/3)\sin(6\pi t) + (1/5)\sin(10\pi t) + \cdots)$.