18.03 Problem Set 4

Due by 1:00 P.M., Friday, March 17, 2006.

I encourage collaboration in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible**.

II. Second-order linear equations

R7	Th 2 Mar	Second order equations: introduction
L11	F 3 Mar	The spring-mass-dashpot model; superposition; characteristic polynomial, real roots; initial conditions; EP 2.1, 2.2, 2.3 up to "Polynomial Operators"; SN 9.
L12	M 6 Mar	Complex roots; damping conditions: EP 2.3, 2.4.
R8	T 7 Mar	ditto
L13	W 8 Mar	Inhomogeneous equations, superposition: Notes O.1, EP 2.6 (pp. 157–159 only; see SN 7 if you want to learn about beats).
R9	Th 9 Mar	ditto
L14	F 10 Mar	Operators and exponential signals: SN 10, Notes O.1, 2, 4, EP 2.6 (pp. 165–167).
L15	M 13 Mar	Undetermined coefficients: SN 11, EP 2.5 (pp. 144–153).
R10	T 14 Mar	ditto
L16	W 15 Mar	Frequency response: SN 14.
R11	Th 16 Mar	ditto
L17	F 17 Mar	Applications.

Part I.

13. (W 8 Mar) Notes: 2C-1: c-e.

14. (F 10 Mar) None.

15. (M 13 Mar) None.

16. (W 15 Mar) [Recitation 11 problem; see also the Mathlet `Amplitude and Phase: First Order`] Express the amplitude of the sinusoidal system response to $\dot{x} + x = \cos(\omega t)$ as a function of ω. Sketch the graph of this function of ω (for $\omega \geq 0$). At what circular frequency is the amplitude of the system response just half that of the input signal? What is the value of ω for which the phase lag is $\pi/4$?

Part II.

13. (W 8 Mar) [Complex roots, damping conditions] This problem will use the Mathlet `Damped Vibration`, available in the tools section. Open this applet. It's about solutions of the second order homogeneous equation $\ddot{x}+b\dot{x}+kx=0$. The initial conditions are set using the box at left. In it the horizontal direction gives $\dot{x}(0)$ and the vertical direction gives $x(0)$. The right graphing window displays the corresponding solution.

Move the cursor around in that box and observe the behavior of the left end of the graph in the right window. Verify for yourself that the slope increases when the horizontal slider is moved right, and that the value increases when the vertical slider is moved up.

(a) Now set $b = 0.00$, $k = 2.50$. What is the differential equation we are looking at? Pick any value of the initial conditions you want (except $(0, 0)$). State them, and measure the horizontal (time) coordinate of the first and second maxima of the solution. Their difference is the period, P. On the other hand, what is the computed period of the general (nonzero) solution of the ODE? Do they agree?

(b) Now let's introduce some damping: set $k = 2.50$ and $b = 0.50$ The solution is of the form $x = Ae^{-bt/2}\cos(\omega_d t - \phi)$, for some choice of amplitude A and phase lag ϕ which depend upon the initial conditions. $b = 0.50$ for us. The circular frequency ω_d is called the "damped frequency." What is ω_d? (You have to solve for it using the theory.) What then is the quasiperiod $P = 2\pi/\omega_d$? What effect has adding a dashpot had on the period of oscillation of this system?

Now, in the Mathlet, choose initial conditions as you like (except $(0, 0)$). State them, and measure the coordinates of the first, second, and third maxima of the solution: $(t_1, x_1), (t_2, x_2), (t_3, x_3)$. You may have to zoom in or out using the powers of ten buttons to get an accurate view of where the maxima occur. Compute the differences $t_2 - t_1$ and $t_3 - t_2$, and the ratios x_2/x_1 and x_3/x_2. Do the time differences match your computation of P?

(c) From the formula, $x(t) = 0$ just where $\cos(\omega_d t - \phi) = 0$, and these are spaced out in intervals of $P/2$. This is independent of the value of b. But it's not so clear that the maxima of $x(t)$ are spaced the same way that the maxima of $\cos(\omega_d t - \phi)$ are. They certainly don't occur at the same time that the maxima of $\cos(\omega_d t - \phi)$ do if $b \neq 0$. How do I know that?

(d) But if your measurements were made accurately, you should find that the time interval between successive maxima seems to equal the quasi-period (and so independent of initial condition). Verify this observation by computation: that is, show that the difference between successive maxima of the function $e^{-bt/2}\cos(\omega_d t - \phi)$ is P. (Do this by setting the derivative to zero. You will want to say how much you have to increase t before a given value of $\tan(\omega_d t - \phi)$ re-occurs. Remember, maxima and minima alternate.)

(e) Now use what you learned in **(d)** to show that the ratio of one maximum to the prior one is independent of which maximum you chose, and also independent of the choice of initial conditions. How well does it match with what you measured in **(b)**?

13. (f) (Not related to the above.) For each of the following homogeneous linear equations, find the appropriate number of independent real solutions, and write down the general real solution.

(i) $\dot{x} + 2x = 0$

(ii) $\ddot{x} + 5\dot{x} + 4x = 0$

(iii) $\dfrac{d^3x}{dt^3} + x = 0$

14. (F 10 Mar) [Inhomogeneous equations; ERF]

> **Exponential Response Formula:** If $p(s)$ is the characteristic polynomial for a linear constant coefficient equation (of any order) with input signal Ae^{rt}, then $Ae^{rt}/p(r)$ is a solution (provided $p(r) \neq 0$). See SN §10 or Notes O.4.

(a) Find a particular solution of:

(i) $\dot{x} + 2x = 2e^{-3t}$

(ii) $\ddot{x} + 5\dot{x} + 4x = e^{-3t}$

(iii) $\dfrac{d^3x}{dt^3} + x = e^{-3t}$

(b) For each of the following ODEs, write down the sinusoidal solution. Do this by replacing the equation by a new one having exponential signal, solving it, and finding the real or imaginary part.

(i) $\dot{x} + 2x = 2\sin(3t)$.

(ii) $\ddot{x} + 5\dot{x} + 4x = \cos(2t)$

(iii) $\dfrac{d^3x}{dt^3} + x = \cos(2t)$.

15. (M 13 Mar) [Undetermined coefficients, Superposition]

(a) Find a particular solution of:

(i) $\dot{x} + 2x = t^2$.

(ii) $\ddot{x} + 5\dot{x} + 4x = 4 + 5t^2$.

(iii) $\dfrac{d^3x}{dt^3} + x = 1 + t^3$

(b) Write down the general solution of each of the following ODEs.

(i) $\dot{x} + 2x = 2e^{-3t} + 4\sin(3t)$

(ii) $\ddot{x} + 5\dot{x} + 4x = e^{-3t} + 4 + 5t^2$

(iii) $\dfrac{d^3x}{dt^3} + x = 1 + t^3 + 4\cos(2t)$

16. (W 15 Mar) [Frequency response] This problem will use the Mathlet `Amplitude and Phase: Second Order`. Open the applet and play around with it. Animate the spring system by pressing the [>>] key. Chose various values of b, k, and ω, and watch the results.

> **Justification of the equation written at the top of the screen.** The driving force, F_{ext}, is provided by motion of the plunger at the top. Suppose that the length of the spring when it is relaxed is d. Measure the position of the plunger with reference to a chosen zero point, and measure the position of the mass with reference to a zero point which is d units lower. This means that $x = y$ when the spring is relaxed, exerting no force. Now the force on the mass is given by $-b\dot{x} - k(x - y)$, so, if we assume that $m = 1$, we have the equation $\ddot{x} + b\dot{x} + kx = ky$. This Mathlet pictures the situation in which the plunger moves sinusoidally with amplitude 1, centered around $y = 0$, and it sets the clock so that $y = \cos(\omega t)$.

(a) There is a white line segment on the main window, joining a blue diamond to a yellow diamond. What does its length signify? What does it mean when the blue dot is above the yellow one? Below? when they coincide?

Set $k = 4.00$ and $b = 0.50$. Slide ω through its range of values and watch how the system response varies.

(b) For what value of ω is the gain, or the amplitude of the system response, the greatest? You can discriminate better among the top values by clicking the [Bode and Nyquist Plots] button and then rolling the cursor over the top right window. This causes a yellow readout giving the amplitude.

The three new windows plot the gain and the phase lag of the system response as functions of the circular frequency of the input signal ω. (Actually $-\phi$ is graphed rather than ϕ.) Verify for yourself that for various values of ω the values of A and ϕ look right for the displayed graph of x. You can get numerical readouts by positioning the cursor over either of these windows.

The lower right window displays the complex number $k/p(i\omega)$. The Exponential Response Formula shows that $\ddot{z} + b\dot{z} + kz = ke^{i\omega t}$ has exponential solution $z_p = (k/p(i\omega))e^{i\omega t}$ where $p(s) = s^2 + bs + k$ is the characteristic polynomial. The modulus of $k/p(i\omega)$ is the amplitude of the sinusoidal system response (k times the "gain") and the argument of $k/p(i\omega)$ is the negative of the phase lag.

(c) When ω is set at the resonant peak which you found in **(b)**, the phase lag seems to be very close to $\pi/2 = 90°$. What are the actual values of ω and A when the gain is maximal? Since the amplitude peak represents "resonance," let's write ω_r and $A(\omega_r)$ for these values. The first step will be to differentiate $f(\omega) = |p(i\omega)|^2$ with respect to ω. You'll need a calculator to get an approximate value (to 3 decimal places) of ω_r.

(d) Explain why the phase lag is exactly $\pi/2$ just when $\operatorname{Re} p(i\omega) = 0$, and calculate what ω must be for this to occur in the general case $\ddot{x} + b\dot{x} + kx = k\cos(\omega t)$. Does this value of ω coincide with the value giving the largest gain in the example we have been studying?