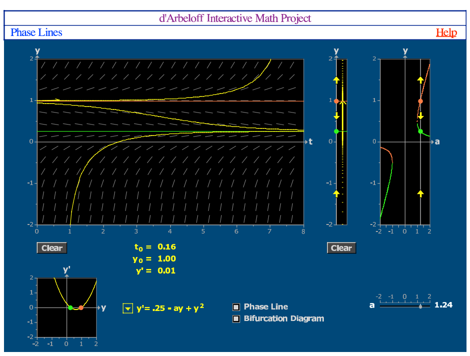
18.03 Problem Set 3 Solutions: Part II

Each problem is worth 16 points, spread across Parts I and II. The Part I problems are worth: **8.** There are 8 parts, each worth 1 point. **9.** 2 points. **11.** 5 points (2-C1(a) = EP 2.1: 33!). **12.** 4 points.

8. **(a)** [2] The Mathlet shows a semi-stable critical point when $a = 1.00$. The critical points of $\dot{y} = .25 - ay + y^2$ occur at values of y given by the solutions of $0 = .25 - ay + y^2$, which are $(a/2) \pm \sqrt{a^2 - 1}/2$. The smallest positive value of a for which the expression inside the square root is not negative is $a = 1$, so for all smaller values of a the solutions grow without bound. When $a = 1$, the equation is $\dot{y} = .25 - y + y^2 = (.5 - y)^2$, so the derivative is always non-negative and the critical point (at $y = .5$) is semi-stable. Thus the slightest deviation of lovebug population above that value gets into the region in which the population continues to grow, without bound. So this level of spray is not safe for Farmer Jones.

(b) [2] To obtain a critical point at $y = 0.25$ we want a to be such that $0 = .25 - ay + y^2$ for that value of y: $0 = (1/4) - a(1/4) + 1/16$, or $a = 5/4$. This is confirmed by the Mathlet (though you have to accept $a = 1.24$ as an approximation to $a = 5/4$). As long as the initial population is at most the value of the upper critical point, the spray will force the population down to $y = .25$. The values of y for which $.25 - (1.25)y + y^2 = 0$ are $y = .25$ and $y = 1$, which is very close to what the Mathlet shows.



(c) [2] This problem is slightly badly worded, since solutions above the equilibrium $y = 1$ reach $+\infty$ in finite time, while solutions below the equilibrium reach $-\infty$ in finite time as time is run backwards.

(d) [2] The bifurcation diagram plots the critical values of y against the parameter a. In this example, it plots the roots of $.25 - ay + y^2$ against a. So it is given by the equation $.25 - ay + y^2 = 0$.

9. **(a)** [5] $\dot{y} = .25 + y^2$ is separable: $\dfrac{4\,dy}{1 + 4y^2} = dt$. With $u = 2y$, $du = 2\,dy$, and $2\arctan(2y) = t - c$. The initial condition gives $c = 0$, so $y = (1/2)\tan(t/2)$. This becomes infinite for the first time after $t = 0$ when $t/2 = \pi/2$ or $t = \pi$.

(b) [5] $\dot{x} = Ix + q$ has a critical point at $x = -q/I$, which is unstable because of the assumption that $I > 0$. If $q < 0$ then $-q/I > 0$. This is the "retirement" scenario: you keep a certain amount of money in the bank and withdraw exactly the interest payments and nothing more. If $q > 0$ the $-q/I < 0$. This is the "credit card" scenario: you owe the bank a certain amount, and they charge you interest for it, which you pay without ever reducing or increasing the amount you owe.

(c) [4] Using $x(t) = 1$, the equation $\dot{x} + p(t)x = q(t)$ gives $p(t) = q(t)$. Using $x(t) = e^{-t}$, it gives $-e^{-t} + p(t)e^{-t} = q(t) = p(t)$, which can be solved to get $p(t) = e^{-t}/(e^{-t} - 1)$.

11. (a) [1] Looks like about -0.75.

(b) [5] The ODE is $\ddot{x}+2\dot{x}+3/4 = 0$. The characteristic polynomial is $s^2+2s+3/4$, and by the quadratic formula the roots are $-1 \pm \sqrt{1-(3/4)} = -1 \pm (1/2)$, or $-1/2$ and $-3/2$. Both roots are negative: this is the overdamped situation. The general solution is $c_1e^{-t/2} + c_2e^{-3t/2}$. Evaluate at $t = 0$ to get $1/2 = x(0) = c_1 + c_2$ or $c_2 = 1/2 - c_1$. Differentiate to get $\dot{x}(0) = -c_1/2 - (3/2)c_2 = -c_1/2 - (3/2)(1/2 - c_1) = c_1 - 3/4$ or $c_1 = 3/4 + \dot{x}(0)$ and $c_2 = 1/2 - (3/4 + \dot{x}(0)) = -1/4 - \dot{x}(0)$. So the solution is $x = (3/4 + \dot{x}(0))e^{-t/2} - (1/4 + \dot{x}(0))e^{-3t/2}$.

(c) [5] Solve $x(t) = 0$: $(3/4 + \dot{x}(0))e^{-t/2} = (1/4 + \dot{x}(0))e^{-3t/2}$. Multiply through by $e^{t/2}$: $3/4 + \dot{x}(0) = (1/4 + \dot{x}(0))e^{-t}$ or $e^{-t} = (3/4 + \dot{x}(0))/(1/4 + \dot{x}(0))$. This occurs for some $t > 0$ as long as the right hand side is between 0 and 1. So numerator and denominator have to have the same sign. They do if $\dot{x}(0) > -1/4$ or if $\dot{x}(0) < -3/4$. If $\dot{x}(0) > -1/4$ however then both numerator and denominator are positive, so, since the numerator is greater than the denominator, the quotient is greater than 1. This shows that $x(t) = 0$ for some $t > 0$ exactly when $\dot{x}(0) < -3/4$: $v = -3/4$, in agreement with what the Mathlet showed. [Since there's then only one value of t giving $x = 0$, this work shows that the door swings through at most once for $t > 0$. If $\dot{x}(0) > -1/4$, then you get exactly one value of t for which $x(t) = 0$ but that value of t is negative. If $-3/4 \leq \dot{x}(0) \leq -1/4$, the solution never becomes zero in the past or in the future.]

When $\dot{x}(0) = -3/4$, the solution is $(1/2)e^{-3t/2}$.

12. (a) [4] $\cosh(0) = 1$ and $\sinh(0) = 0$. $\cosh'(x) = \sinh(x)$ and $\sinh'(x) = \cosh(x)$, so $\cosh'(0) = 0$ and $\sinh'(0) = 1$. Therefore the value of $y = c_1\cosh(x) + c_2\sinh(x)$ at $x = 0$ is $y(0) = c_1$, and the value of $y' = c_1\cosh'(x) + c_2\sinh'(x)$ at $x = 0$ is $y'(0) = c_2$: $y = y(0)\cosh(x) + y'(0)\sinh(x)$.

(b) [4] Let's write $y_1(x) = \cos(\omega x)$ and $y_2(x) = \sin(\omega x)$. Then $y_1' = -\omega y_2$ and $y_2' = \omega y_1$, so $y_1'' = -\omega^2 y_1$ and $y_2'' = -\omega^2 y_2$. Then $y_1(0) = 1$, $y_2(0) = 0$, $y_1'(0) = 0$, $y_2'(0) = \omega$, so the value of $y = c_1y_1 + c_2y_2$ at $x = 0$ is $y(0) = c_1$ while the value of $y' = c_1y_1' + c_2y_2'$ at $x = 0$ is $y'(0) = c_2\omega$: $y = y(0)\cos(\omega x) + (y'(0)/\omega)\sin(\omega x)$.

(c) [4] The ODE is $\dfrac{d^3x}{dt^3} - \dfrac{dx}{dt} = 0$. The roots of the characteristic polynomial are $-1, 0, 1$ so there are three exponential solutions, e^{-t}, 1, and e^t. Suppose that $e^{-t} = c_1 + c_2e^t$. As $t \to \infty$, the right hand side grows large unless $c_2 = 0$. Since the left hand side does not grow large, we must have $c_2 = 0$. But then the right hand side is the constant c_1, while e^{-t} falls to 0 as $t \to \infty$. This shows that our supposition must have been wrong. Same with writing 1 or e^t as linear combinations of the other two. There are many other ways to see that these three functions are linearly independent.