18.03 Problem Set 3 - Complete

Due by 1:00 P.M., Friday, March 10, 2006,
[Not Wednesday, as stated on the Syllabus. Problems relating to Wednesday's lecture will be on PS4.]

I encourage collaboration in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible**.

I. First-order differential equations

L8	F 24 Feb	Autonomous equations; the phase line, stability: EP 1.7, 7.1.
L9	M 27 Feb	Linear vs nonlinear: Review.
R6	T 28 Feb	Exam preparation.
L10	W 1 Mar	**Hour Exam I**

II. Second-order linear equations

R7	Th 2 Mar	Second order equations: introduction
L11	F 3 Mar	The spring-mass-dashpot model; superposition; characteristic polynomial; EP 2.1, 2.2.
L12	M 6 Mar	Real roots, initial conditions: EP 2.3, SN 9.
R8	T 7 Mar	ditto
L13	W 8 Mar	Complex roots; damping conditions: EP 2.4.
R9	Th 9 Mar	ditto
L14	F 10 Mar	Inhomogeneous equations, superposition: Notes O.1, EP 2.6 (pp. 157–159 only; see SN 7 if you want to learn about beats).

Problems **8**, **9**, and **10** are identical to what was handed out last week. Problems **11** and **12** are new.

Part I.

8. (F 24 Feb) Notes: 1E-1.

9. (M 27 Feb) [Recitation 6 problem] Find the sinusoidal solution of $\dot{x} + 2x = \cos(2t)$ in polar form, $A\cos(\omega t - \phi)$, in the following way: First find the exponential solution of the corresponding equation with complex exponential right hand term; it is $z_p = \dfrac{1}{2i+2}e^{2it}$. Then find A and ϕ such that $\dfrac{1}{2i+2} = Ae^{-i\phi}$, and use this to re-express $z_p = Ae^{(2t-\phi)i}$. Now take the real part.

10. (W 1 Mar) Nothing

11. (F 3 Mar) EP 2.1: 1, 3, 43; Notes: 2-C1 a,b.

12. (M 6 Mar) EP 2.3: 1, 2, 21, 28.

Part II.

8. (F 24 Feb) [Autonomous Equations] This problem will use the Mathlet `Phase Lines`. Open the applet and understand its use and conventions. Click on [Phase Line] to see a representation of the phase line. Note the color coding: a green dot represents a stable or attracting equilibrium; red represents an unstable or repelling equilibrium; and blue represents a "semi-stable" equilibrium.

The autonomous ODE $\dot{y} = .25 - ay + y^2$ models a population of Australian lovebugs, which infest pomagranites in Farmer Jones's orchard. y is measured in megabugs, or millions of bugs. The term .25 reflects a constant immigration into Mr Jones's orchard from the neighboring orchards (where the pomagranites are inferior). These pests can be kept in check using an expensive bioengineered spray. Application at a rate a moves the natural rate of growth of the lovebug population from y (corresponding to $\dot{y} = y^2$) down to $y - a$ (corresponding to $\dot{y} = (y - a)y$).

(a) Use the Mathlet to determine (approximately) the smallest rate a which will contain the lovebug population at a finite level provided that it starts at a sufficiently low level. What is that level? Now check this answer analytically. Why is this level of application a dangerous strategy for Farmer Jones?

(b) Better will be a choice of a which brings the lovebug population down to $y = 0.25$. What rate a will lead to that result, according to the Mathlet? How large an initial population will this rate of application control? Now check these answers analytically.

(c) For this value of a, there are five different behaviors possible for the lovebug population. Two solutions exhibit the "same behavior" if one is a time-translate of the other. Sketch one solution of each of the five types. Your sketch should make it clear what the behavior of the solution is as $t \to -\infty$ and as $t \to \infty$.

(d) Invoke the Bifurcation Diagram for this autonomous equation. Move a along its slider to see the variety of behaviors the phase line of $\dot{y} = .25 - ay + y^2$ as a varies. The green and red curve in the newly displayed bifurcation plane represents the equilibrium points for those equations, for various values of a. Give an equation for that curve.

9. (M 27 Feb) [Linear vs Nonlinear] (a) This continues problem **8.** In the absence of the spray, $\dot{y} = .25 + y^2$. Solve this equation analytically subject to the initial condition $y(0) = 0$ (so this is virgin lovebug territory at $t = 0$). At what value of t does the lovebug population become infinite?

(b) Consider the bank account equation $\dot{x} - Ix = q$ in the case when the interest rate I and the rate of savings q are both constant. This is then an autonomous equation. Assume $I > 0$. Sketch the phase line. If $q < 0$, what is the sign of the equilibrium? Please describe a scenario that this represents. What about when $q > 0$?

(c) Suppose it is known that the constant function with value 1 and the function e^{-t} are both solutions of the linear equation $\dot{x} + p(t)x = q(t)$. What are the functions $p(t)$ and $q(t)$?

10. (W 1 Mar) [Hour Exam: Nothing but good luck]

11. (F 3 Mar) [Second order models, real roots, initial conditions] Saloon doors can swing through the door frame. A good door damper will slow a swinging door down so it does not swing through the door frame—unless you shove the door hard toward the frame. In that case, it will swing through and return from the other side. This is a characteristic of "overdamping."

This problem will study this effect, using the Mathlet `Damped Vibration`. Open the applet. It's about solutions of the second order homogeneous equation $\ddot{x}+b\dot{x}+kx = 0$. The initial conditions are set using the box at left. In it the horizontal direction gives $\dot{x}(0)$ and the vertical direction gives $x(0)$. The right graphing window displays the corresponding solution.

Move the cursor around in the initial conditions box and observe the behavior of the left end of the graph in the right window. Verify for yourself that the slope increases when the horizontal coordinate increases, and that the value increases when the vertical coordinate increases. These coordinates can also be controlled using the sliders along the edge of the initial conditions box.

As you decrease the damping constant b, you will see that the sytem becomes "underdamped": the solutions oscillate. In this problem, though, we will study the case in which $b = 2$ and $k = 0.75$. (The applet won't let you choose $k = 0.75$. Approximate it by 0.74.) Check that this is overdamped: the characteristic polynomial has two distinct real roots, and the solutions don't oscillate.

Now set $x(0) = 0.50$. Set the $\dot{x}(0)$ slider at -1.00 and start to increase it slowly. Watch the effect on the solution curve. At first it swings through $x = 0$, but soon seems not to. You can get a better picture by zooming in using one of the power-of-ten buttons. Work with this till you come up with the smallest value of $\dot{x}(0)$ which does not result in a solution crossing the t axis. (It will be a negative number.)

(a) State the value you discover.

(b) Now find the general solution of this ODE, with these values of k and b. Express the constants of integration in terms of $\dot{x}(0)$ (using $x(0) = 0.5$).

(c) Finally, verify that there is a number v such that the solution never takes on the value zero for $t > 0$ if $\dot{x}(0) \geq v$, while it does if $\dot{x}(0) < v$. Does v agree with your measurement on the Mathlet? What is the solution with $\dot{x}(0) = v$?

12. (M 6 Mar) [Real roots, initial conditions, characteristic polynomial] The hyperbolic cosine and sine functions are defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Since the roots of the characteristic polynomial of $y'' - y = 0$ are ± 1, e^x and e^{-x} are solutions. By superposition, so are $\cosh(x)$ and $\sinh(x)$.

(a) In fact, any solution of $y'' - y = 0$ can be expressed as a linear combination of $\cosh(x)$ and $\sinh(x)$ as well. Express the coefficients c_1 and c_2 in $y(x) = c_1 \cosh(x) + c_2 \sinh(x)$ in terms of the initial conditions $y(0)$ and $y'(0)$.

(b) Check that $\cos(\omega x)$ and $\sin(\omega x)$ are solutions of $y'' + \omega^2 y = 0$. [This is another of those facts which you should memorize!] Write a general solution y of $y'' + \omega^2 y = 0$ as a linear combination of them. Express the coefficients c_1 and c_2 in $y = c_1 \cos(\omega x) + c_2 \sin(\omega x)$ in terms of $y(0)$ and $y'(0)$.

For example, take $\omega = 1$ and write down the solution with $y(0) = 5$, $y'(0) = 3$.

(c) Write down the constant coefficient homogeneous linear differential equation with characteristic polynomial $p(s) = s^3 - s$, and find three linearly independent solutions for it. Show that they are linearly independent by assuming that the third is a linear combination of the other two and then seeing that this can't happen. One way to see that it can't happen, in this case, is to think about what happens when $t \to \infty$.