

18.03 Problem Set 1

Due by 1:00 P.M., Wednesday, February 15, 2006.

Part I of each problem set will consist of problems which are either rather routine, or for which solutions are available in the back of the book, or in 18.03 *Notes and Exercises*.

Part II contains more challenging and novel problems. They will be graded with care (Complain if they are not!) and contribute the bulk of the Homework grade. They will help you develop an understanding of the material.

Problems in both parts are keyed closely to the lectures, and numbered to match them. Try the problems as soon as you can after the indicated lecture.

Each day counts 16 points. Usually the Part II problems will be given more credit than the Part I problems. The problem sets vary in length, because they cover either three or four lectures, so the maximum score is either 48 or 64. This PS is worth 64 points. Exceptionally, it includes a problem related to Tuesday's recitation.

I encourage collaboration in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible.**

I. First-order differential equations

R1	T 7 Feb	Simple models and separable equations: EP 1.1, 1.4.
L1	W 8 Feb	Direction fields, existence and uniqueness of solutions: EP 1.2, 1.3; Notes G.1; SN §1.
R2	Th 9 Feb	Isoclines; Use of fundamental theorem of calculus: Notes G.1, D.
L2	F 10 Feb	Numerical methods: EP 6.1, 6.2; Notes G.2.
L3	M 13 Feb	Linear equations; models: EP 1.5; SN §2.
R3	T 14 Feb	Ditto
L4	W 15 Feb	Solution of linear equations; variation of parameter: EP 1.5, SN §3.

Part I.

0. (T 7 Feb) EP 1.4: 39, 66; Notes 1A-5c; EP 1.4: 1, 9, 21. (For problems EP 1.4: 1 and 9, explain carefully how the constant of integration comes to be where it is and how it comes to take on arbitrary values.)

The problem EP 1.4:66 cannot be found in the 4th edition of the book. Here it is:

Early one morning it began to snow at a constant rate. At 7 A.M. a snowplow set off to clear a road. By 8 A.M. it had traveled 2 miles, but it took two more hours (until 10 A.M.) for the snowplow to go an additional 2 miles.

(a) Let $t = 0$ when it began to snow and let x denote the distance traveled by the snowplow at the time t . Assuming that the snowplow clears snow from the road at a constant rate (in cubic feet per hour, say) show that

$$k \frac{dx}{dt} = \frac{1}{t}$$

where k is a constant.

(b) What time did it start snowing? (Answer: 6 A.M.)

1. (W 8 Feb) Notes 1C-1abe.

2. (F 10 Feb) IC-4.

3. (M 13 Feb) EP 1.5: 16, 38.

Part II.

0. (T 7 Feb) An airplane of mass m is propelled through the air in a straight line with thrust T . The air it flies through creates a drag, which is proportional to the square of the velocity. By $F = ma$, the velocity v satisfies the differential equation

$$m \frac{dv}{dt} = T - bv^2.$$

For definiteness, take $T/m = 9$ and $b/m = 1$.

(a) Find the general solution to this equation.

(b) Find the particular solution with $v(0) = 0$.

(c) Is there a “terminal,” or limiting, velocity? If so, what is it?

(d) Why is the equation physically unrealistic when $v < 0$?

1. (W 8 Feb) In this problem you will study solutions of the differential equation

$$\frac{dy}{dx} = y^2 - x^2.$$

Solutions of this equation do not admit expressions in terms of the standard functions of calculus, but we can study them anyway using the direction field.

(a) Draw a large pair of axes and mark off units from -4 to $+4$ on both. Sketch the direction field given by our equation. Do this by first sketching the isoclines for slopes $m = -2$, $m = 0$, and $m = 2$. On this same graph, sketch a couple of solutions.

Having done this, we will continue to investigate this equation using one of the Mathlets. So go to [tools section](#) and select **Isoclines** from the menu. (If you are using a Mac, you may find it difficult to use Firefox for these applets and should probably pick a different browser.) Play around with this applet for a little while. The Mathlets have many features in common, and once you get used to one it will be quicker to learn how to operate the next one. Clicking on "Help" pops up a window with

a brief description of the applet's functionalities.

Select $y' = y^2 - x^2$ from the pull-down menu, and verify your work in **(i)**. (Nothing to turn in here.)

By clicking on the graphing window, cause several solutions to be drawn. It appears that solutions fall into two main types, according to their behavior as x becomes large: some increase without bound, while others decrease without bound.

(b) It appears that there is some constant y_0 such that if y is a solution with $y(0) > y_0$ then y becomes large as x becomes large, while if $y(0) < y_0$ then y decreases as x increases. By experimenting with the applet, find y_0 to within 0.01: that is, find two numbers which differ by 0.02 and have y_0 between them. Explain briefly what you did.

(c) Is there some function $f(x)$ such that $y(x) > f(x)$ for all $x > 0$ whenever y is a solution with $0 < y(0) < y_0$ (i.e., which is falling for sufficiently large x)? You can use the Mathlet to get some ideas, but then you should find a way to explain the answer you come up with.

(d) Suppose that a solution y has a maximum at the point (a, b) . What can you say about the relationship between a and b ? Explain.

2. (F 10 Feb) (a) The solution to $y' = y$ with $y(0) = 1$ is $y = e^x$. Compute y_k for $k = 0 \dots 3$ for n fixed but arbitrary. What is the Euler approximation for $e = y(1)$, using n equal steps?

The rest of this problem will use another Mathlet to investigate the dependence of the error of Euler's method on the step size. So invoke **Euler's Method** and familiarize yourself with it for a few minutes. It illustrates Euler's method applied to the differential equation $y' = f(t, y)$, where $y' = dy/dt$. Select **f(t,y)=y sin(t)**, and carefully select the initial condition $(t, y) = (0.00, 1.00)$. Use the tool to obtain the Euler's method estimates of $y(1)$, for the various step sizes available, and also the "actual" value.

(b) Make a table with the following columns: the step size h , the Euler estimate using this step size, the "error" corresponding to h (that is, the actual value minus the estimated value), and the error divided by h . Does your table support the claim that the error is approximately proportional to h ? For comparison, make a table of the error divided by h^2 . Which is closer to constant? [The "improved Euler method," or RK2, would lead to errors roughly proportional to h^2 . RK4 would give errors roughly proportional to h^4 .]

(c) Is the estimated value larger or smaller than the actual value? Please explain why this is, in terms of the direction field.

(d) Find the solution to $y' = y \sin(t)$ with $y(0) = 1$ analytically, and use a calculator to find $y(1)$. Does it coincide with the "Actual" value given by the Mathlet?

3. (M 13 Feb) A *mortgage* is a type of loan in which the borrower pays installments each of which partly pays down the debt and partly pays interest on the balance owed. Suppose the interest rate is fixed, say I in units of $(\text{years})^{-1}$ (so $I = 0.05$ means 5% per year), and the duration of the mortgage is T years (so it should be entirely paid off after exactly T years). Suppose the amount of money borrowed is M (in dollars). Write q for the rate of payment, measured in dollars per year, and assume that it

is constant. By this we mean that the borrower is paying continuously, rather than in monthly payments. This is a typical mathematical approximation, replacing a discrete problem with a continuous one. It will give quite good agreement in this case.

(a) Model this process by a differential equation. Explain your steps. You will have to include q as an unknown constant; its value will emerge at the end of this problem.

(b) Then find the general solution to this differential equation.

(c) Finally, find the particular solution resulting in a loan balance of 0 at time T . Use this to solve for q in terms of I , T , and M .

(d) A typical mortgage has $T = 30$ and $I = 0.05$. Estimate the *monthly* payments if $M = 10^5$. How many dollars do you end up paying to the bank for each dollar you borrow, over the entire life of the loan?