Independence Ratios of Nearly Planar Graphs

Mike Albertson
Smith College

Are there intuitive relaxations of planarity that support a lower bound on the independence ratio?

Sometimes the independence ratio is more fun to look at than the chromatic number.

Outline

1. Introduction: the independence ratio
2. Embedded graphs
    a) Early results
    b) Open questions
3. Graphs with given thickness
4. Graphs with given crossing number
5. Independence questions for new varieties of nearly planar graphs

The Independence Ratio

The Fraction [V65]; The Name [AH75]

Suppose G is a graph with n vertices. Let
$\alpha(G) = \max\{|U| : U \subseteq V(G);\ x, y \in U \Rightarrow xy \notin E(G)\}$.
The *independence ratio*, (“$\mu(G)$”), is defined by

$$\mu(G) = \frac{\alpha(G)}{n} \,.$$

Since a color class is independent, $\alpha(G) \geq \frac{n}{\chi(G)}$.
Thus $\mu(G) \geq \frac{1}{\chi(G)}$.

There is a circular refinement *viz.* $\mu(G) \geq \frac{1}{\chi_c(G)}$.

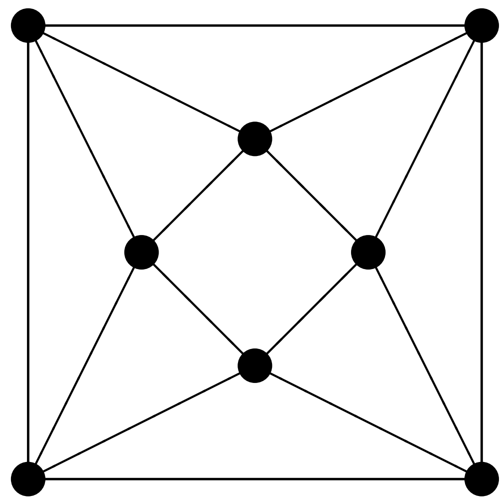
Planar Graphs

Conj [EV 60s] If G is planar, then $\mu(G) \geq \frac{1}{4}$.

Planar Graphs

Conj [EV 60s] If G is planar, then $\mu(G) \geq \frac{1}{4}$.

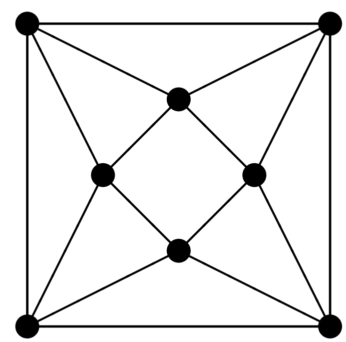
EV is sharp



Planar Graphs

Conj [EV 60s] If G is planar, then $\mu(G) \geq \frac{1}{4}$.

EV is sharp



Th [A74] If G is planar, then $\mu(G) > \frac{2}{9}$.

4CT $\Rightarrow$ EV {still no independent proof}

Embedded Graphs (we know a lot)

Let S_g denote the orientable surface with g handles.

Th [H91] If G is embedded on S_g, then $\chi(G) \leq H(g) = \lfloor \frac{7+\sqrt{48g+1}}{2} \rfloor$. Thus $\mu(G) \geq \frac{1}{H(g)}$.

Cor If G is toroidal, then $\mu(G) \geq \frac{1}{7}$.

Th [RY68] $K_{H(g)}$ embeds on S_g .

Th [AH75] Suppose $G \neq K_7, K_6, K_7 \cup K_4,$ or C_{11}^{3} .

If G is toroidal, then $\mu(G) \geq \frac{1}{5}$.

Th [AH75] Suppose $G \neq K_7, K_6, K_7 \cup K_4,$ or C_{11}^3 .
If G is toroidal, then $\mu(G) \geq \frac{1}{5}$.

Th [AH75] Suppose G is toroidal. Given
$\epsilon > 0, \exists\ N(\epsilon)$: if $n > N(\epsilon),$ then $\mu(G) > \frac{2}{9} - \epsilon$.

Th [AH75] Suppose $G \neq K_7, K_6, K_7 \cup K_4,$ or C_{11}^3 .
If G is toroidal, then $\mu(G) \geq \frac{1}{5}$.

Th [AH75] Suppose G is toroidal. Given
$\epsilon > 0, \exists\ N(\epsilon)$: if $n > N(\epsilon),$ then $\mu(G) > \frac{2}{9} - \epsilon$.

Th [AH75] Suppose G embeds on S_g. Given
$\epsilon > 0, \exists\ N(\epsilon, g)$: if $n > N(\epsilon, g),$ then $\mu(G) > \frac{1}{5} - \epsilon$.

Th [AH75] Suppose $G \neq K_7, K_6, K_7 \cup K_4$, or C_{11}^3.
If G is toroidal, then $\mu(G) \geq \frac{1}{5}$.

Th [AH75] Suppose G is toroidal. Given
$\epsilon > 0, \exists\ N(\epsilon)$: if $n > N(\epsilon)$, then $\mu(G) > \frac{2}{9} - \epsilon$.

Th [AH75] Suppose G embeds on S_g. Given
$\epsilon > 0, \exists\ N(\epsilon, g)$: if $n > N(\epsilon, g)$, then $\mu(G) > \frac{1}{5} - \epsilon$.

Th [AH78, S] Suppose G embeds on S. Given
$\epsilon > 0, \exists\ N(\epsilon, S)$: if $n > N(\epsilon, S)$, then $\mu(G) > \frac{1}{4} - \epsilon$.

On any given S only a few graphs have $\mu << \frac{1}{4}$.

Sketch of Proof Technique: Cycle $C \subset G$ embedded on S, is *n.c.* if it is not homotopic to a point.

Width of G [AH75] $w(G) = \min\{|C| : C \text{ is n.c.}\}$.

On any given S only a few graphs have $\mu << \frac{1}{4}$.

Sketch of Proof Technique: Cycle $C \subset G$ embedded on S, is *n.c.* if it is not homotopic to a point.

Width of G [AH75] $w(G) = \min\{|C| : C \text{ is n.c.}\}$.

Th [AH78] If G triangulates S, then $w(G) \leq \sqrt{2n}$.

Cor If G is embedded on S, then $\exists\, U \subset V(G) : |U|$ is small and $G[V - U]$ is planar.

Questions for Embedded Graphs [AH74]

Background:

Th [AS82] G toroidal and $w(G) > 3 \Rightarrow \chi(G) \leq 5$.

Cor If G is toroidal, then $\mu(G) \geq \frac{1}{5} - \frac{3}{5n}$.

Questions for Embedded Graphs [AH74]

Background:

Th [AS82] G toroidal and $w(G) > 3 \Rightarrow \chi(G) \leq 5$.

Cor If G is toroidal, then $\mu(G) \geq \frac{1}{5} - \frac{3}{5n}$.

Conj G toroidal $\Rightarrow \mu(G) \geq \frac{1}{4} - \frac{3}{4n}$.

Questions for Embedded Graphs [AH74]

Background:

Th [AS82] G toroidal and $w(G) > 3 \Rightarrow \chi(G) \leq 5$.

Cor If G is toroidal, then $\mu(G) \geq \frac{1}{5} - \frac{3}{5n}$.

Conj G toroidal $\Rightarrow \mu(G) \geq \frac{1}{4} - \frac{3}{4n}$.

Q ∃? M_S : G embeds on $S \Rightarrow \mu(G) \geq \frac{1}{4} - \frac{M_S}{4n}$

Questions for Embedded Graphs [AH74]

Background:

Th [AS82] G toroidal and $w(G) > 3 \Rightarrow \chi(G) \leq 5$.

Cor If G is toroidal, then $\mu(G) \geq \frac{1}{5} - \frac{3}{5n}$.

Conj G toroidal $\Rightarrow \mu(G) \geq \frac{1}{4} - \frac{3}{4n}$.

Q $\exists$? M_S : G embeds on $S \Rightarrow \mu(G) \geq \frac{1}{4} - \frac{M_S}{4n}$

Q Does $M_{S_g} = 3g$?

So where are we?

G embedded on S is a relaxation of G is planar.

Embedding similar to planarity wrt "μ"-behavior.

So where are we?

G embedded on S is a relaxation of G is planar.

Embedding similar to planarity wrt "μ"-behavior.

Where are we going?

What happens with other relaxations of planarity?

If G is *nearly planar*, what about $\mu(G)$?

Nearly Planar Graphs

- Classic Versions
  - thickness
  - crossing number

- Recent versions
  - locally planar graphs
  - k-quasi-planar graphs
  - k-embedded graphs
  - k-quasi*planar graphs

Thickness (we don't know much)

G is said to have thickness t if G is the union of t planar graphs but no fewer.

Rmks If G has thickness t, then $E \leq t(3n-6)$.
So, if G has thickness t, then $\chi(G) \leq 6t$.
$\exists G$ with thickness t such that $\chi(G) \geq 6t-2$ ($t > 2$).
When $t = 2$, all we know is $9 \leq \chi(G) \leq 12$.

Cor If $t(G) = 2$, then $\mu(G) \geq \frac{1}{12}$.

Th [BH,AG] $t(K_n) = \lfloor \frac{n+7}{6} \rfloor$ ($n \neq 9, 10$)
$t(K_9) = t(K_{10}) = 3$

Independence for Thickness 2 Graphs

- $\exists \mu_2 : t(G) = 2 \Rightarrow \mu(G) \geq \mu_2 \geq \frac{1}{12}$.

Old Conj [A] $\mu_2 = \frac{1}{8}$.

Independence for Thickness 2 Graphs

- $\exists \mu_2 : t(G) = 2 \Rightarrow \mu(G) \geq \mu_2 \geq \frac{1}{12}$.

Old Conj [A] $\mu_2 = \frac{1}{8}$.

NOT! [GS] $\exists G : t(G) = 2, n = 17, \mu(G) = \frac{2}{17}$.

NOT! [GS] $\exists G : t(G) = 2, n = 17, \mu(G) = \frac{2}{17}$.

Independence for Thickness 2 Graphs

- $\exists \mu_2 : t(G) = 2 \Rightarrow \mu(G) \geq \mu_2 \geq \frac{1}{12}$.

Old Conj [A] $\mu_2 = \frac{1}{8}$.

NOT! [GS] $\exists G : t(G) = 2, n = 17, \mu(G) = \frac{2}{17}$.

Open Q What is μ_2?

Q [A] Given $\epsilon > 0, \exists$? $G : t(G) = 2,\ \frac{1}{9} < \mu(G) < \frac{1}{9} + \epsilon$?

Independence for Thickness 2 Graphs

- $\exists \mu_2 : t(G) = 2 \Rightarrow \mu(G) \geq \mu_2 \geq \frac{1}{12}$.

Old Conj [A] $\mu_2 = \frac{1}{8}$.

NOT! [GS] $\exists G : t(G) = 2, n = 17, \mu(G) = \frac{2}{17}$.

Open Q What is μ_2?

Q [A] Given $\epsilon > 0, \exists?\ G : t(G) = 2,\ \frac{1}{9} < \mu(G) < \frac{1}{9} + \epsilon$?

Conj [G] $\mu_2 = \frac{2}{21}$.

Crossing Number (we know even less)

The crossing number of G (cr(G)) is the minimum number of crossings in any drawing of G.

Conj $\mathrm{cr}(K_n) = Z_n = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$.

Th [KMPRS] $\lim_{n\to\infty} \mathrm{cr}(K_n)/Z_n \geq 0.83$.

Q Is $\chi(G)$ bounded by a function of (cr(G))?

Q Is $\mu(G)$ bounded by a function of (cr(G))?

Small Results on Crossings and Colorings [OZ]

Th If cr$(G) \leq$ 2, then $\chi(G) \leq$ 5.

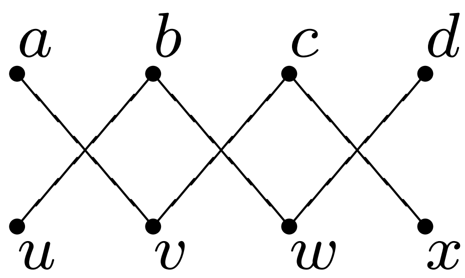
Notation: $\omega(G)$ denotes the *clique number*.

Th If cr$(G) \leq$ 3 and $\omega(G) \leq$ 5, then $\chi(G) \leq$ 5.

Q If cr$(G) \leq$ 5 and $\omega(G) \leq$ 5, is $\chi(G) \leq$ 5?

A Small Result on Crossings and Colorings [A]

Def In a plane graph, two crossings are *dependent* if their eight incident vertices are not distinct.



$\{(av, bu)(vc, bw)\}$ dependent

$\{(av, bu)(cx, dw)\}$ not dependent

Th If G is a plane graph, $\text{cr}(G) \leq 3$, and crossings are independent, then $\chi(G) \leq 5$. Thus $\mu(G) \geq \frac{1}{5}$.

Conj If G is a plane graph and no two crossings are dependent, then $\chi(G) \leq 5$ and $\mu(G) \geq \frac{1}{5}$.

Rmk [A,S] If G is a plane graph and no two crossings are dependent, then $\chi(G) \leq 8$. Thus $\mu(G) \geq \frac{1}{8}$.

Th [A] If G is a plane graph and no two crossings are dependent, then $\chi(G) \leq 6$. Thus $\mu(G) \geq \frac{1}{6}$.

Th [A] If G is a plane graph and no two crossings are dependent, then $\mu(G) \geq \frac{3}{16}$.

Th [A] If G is a plane graph and no two crossings are dependent, then $\mu(G) \geq \frac{3}{16}$.

Pf From crossing independence, $n \geq 4 \cdot \text{cr}(G)$.

Thus $\exists U \subset V(G) : |U| \leq \frac{n}{4}$ & $G[V - U]$ is planar.

$\alpha(G) \geq \alpha(G[V-U]) \geq \frac{1}{4} \cdot \frac{3n}{4} = \frac{3n}{16}$.

Th [A] If G is a plane graph and no two crossings are dependent, then $\mu(G) \geq \frac{3}{16}$.

Pf From crossing independence, $n \geq 4 \cdot \mathrm{cr}(G)$.
Thus $\exists U \subset V(G) : |U| \leq \frac{n}{4}$ & $G[V - U]$ is planar.
$\alpha(G) \geq \alpha(G[V - U]) \geq \frac{1}{4} \cdot \frac{3n}{4} = \frac{3n}{16}$.

Rmk The proof of the μ-result is easier than the proof of the χ-result, but the μ-result is stronger.

A Naive Definition of Locally Planar

Given $x \in V(G)$, let

$N^d[x] = \{u \in V(G) : \text{dist}(x, u) \leq d\}$.

If $G[N^d[x]]$ is planar $\forall x \in V(G)$ and d is large, we could say that G seems locally planar. However,

Th [E59] $\forall k, m \in \mathbb{Z}$ there exists a graph G such that $\chi(G) \geq k$, and the girth of $G \geq m$.

Locally Planar Embedded Graphs

Def Suppose G is embedded on S. If $w(G)$ is large, we say that G is *locally planar*.

Note if $d < \frac{w(G)-1}{2}$, then $\forall x, G[N^d[x]]$ is planar.

The previously mentioned results on the independence ratio justify the above definition.

In addition there are similar coloring results.

Th [H84] If G is embedded on S_g and every edge is short enough, then $\chi(G) \leq 5$.

Th [T93] If G is embedded on S_g and $w(G) \geq 2^{28g+6}$, then $\chi(G) \leq 5$.

Th [DKM05] If G is embedded on S_g and $w(G)$ is large enough, then $\chi_\ell(G) \leq 5$.

A Question on Local Planarity and Thickness

Suppose G is a graph with thickness 2.

For $1 \leq r \leq 4$, does there exist $d = d(r)$:

if $G[N^d[x]]$ is planar, then $\mu(G) \geq \frac{1}{4+r}$?

New Nearly Planar Graphs

Here are recent attempts to capture near planarity.

Some come with extremal results about $|E(G)|$.

For each attempt: **Is there an idea to get from the intuitively attractive definition to a meaningful theorem about μ?**

Another Version of Locally Planar

Def [PPTT02] G is said to be *r-locally planar* if G contains no self intersecting path of length $\leq r$.

Th [PPTT02] $\exists$ 3-locally planar graphs with $E \geq c \cdot n \log(n)$.

Th [PPTT04] If G is 3-locally planar, then $E = \mathrm{O}(n \log(n))$.

Q $\exists$? μ_3 such that every 3-locally planar graph has $\mu(G) \geq \mu_3$?

Q $\exists$? μ_3 such that every 3-locally planar graph has $\mu(G) \geq \mu_3$?

These graphs can have a superlinear number of edges $\Rightarrow$ greedily building an independent set does not work.

Q $\exists$? μ_3 such that every 3-locally planar graph has $\mu(G) \geq \mu_3$?

These graphs can have a superlinear number of edges $\Rightarrow$ greedily building an independent set does not work.

Looking at r-locally planar graphs ($r \geq 4$) gives similar results.

Q $\exists$? μ_3 such that every 3-locally planar graph has $\mu(G) \geq \mu_3$?

These graphs can have a superlinear number of edges $\Rightarrow$ greedily building an independent set does not work.

Looking at r-locally planar graphs ($r \geq 4$) gives similar results.

The examples of r-locally planar graphs with lots of edges have relatively large μ.

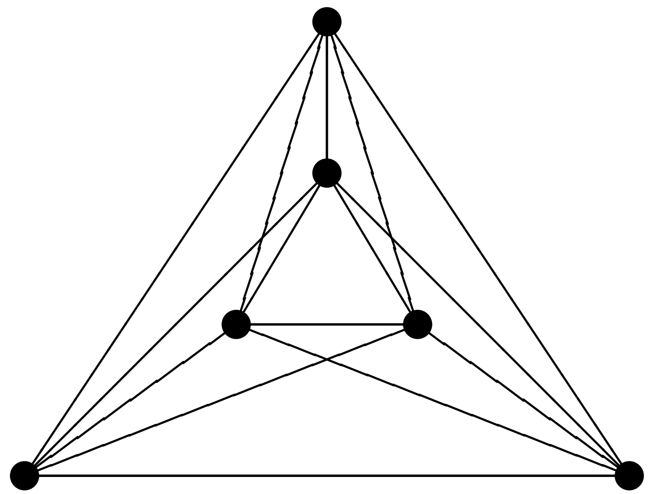
Quasi Planar Graphs

Def [PT97] If, G has a drawing in which no edge crosses more than r other edges, we say that G is *r-quasi planar* (r-q-p).

Th [PT97] If G is r-q-p, then $E \leq (r+3)(n-2)$. Sharp for $0 \leq r \leq 2$ - not close for large r.

Cor If G is r-q-p, then $\mu(G) \geq \frac{1}{2r+6}$.

Th [B84] If G is 1-q-p, then $\chi(G) \leq 6 \Rightarrow \mu(G) \geq \frac{1}{6}$.



The above theorem is sharp

Def [R] Given a planar graph G, the *vertex-face* graph G_{vf} has $V(G_{vf}) = V(G) \cup F(G)$.

$E(G_{vf}) = \{xy : x \text{ is adjacent to or incident with } y\}$.

Def [R] Given a planar graph G, the *vertex-face graph* G_{vf} has $V(G_{vf}) = V(G) \cup F(G)$.

$E(G_{vf}) = \{xy : x$ is adjacent to or incident with $y\}$.

Rmks G planar $\Rightarrow G_{vf}$ is 1-q-p. $K_6 \neq G_{vf}$.

Def [R] Given a planar graph G, the *vertex-face* graph G_{vf} has $V(G_{vf}) = V(G) \cup F(G)$.

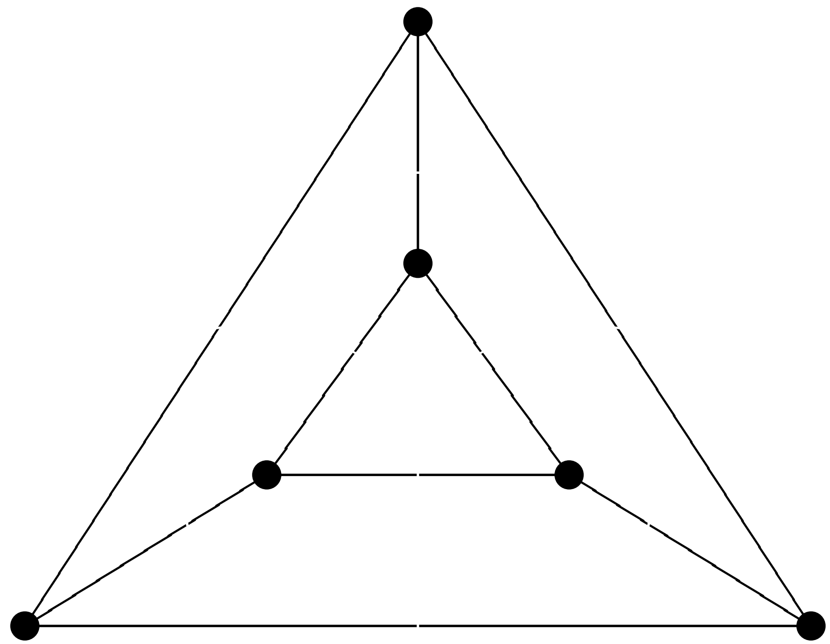
$E(G_{vf}) = \{xy : x$ is adjacent to or incident with $y\}$.

Rmks G planar $\Rightarrow G_{vf}$ is 1-q-p. $K_6 \neq G_{vf}$.

Conj [A] If G is planar, then $\mu(G_{vf}) \geq \frac{2}{11}$.

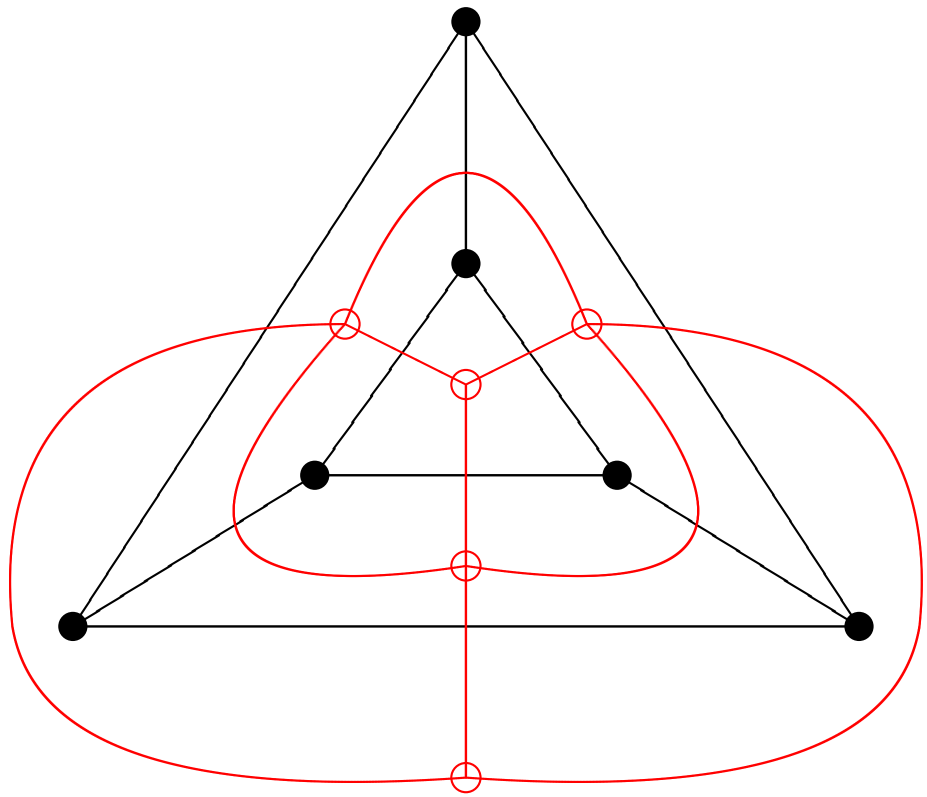
$\mu((K_3 \square K_2)_{vf}) = \frac{2}{11}$.

$K_3 \square K_2$



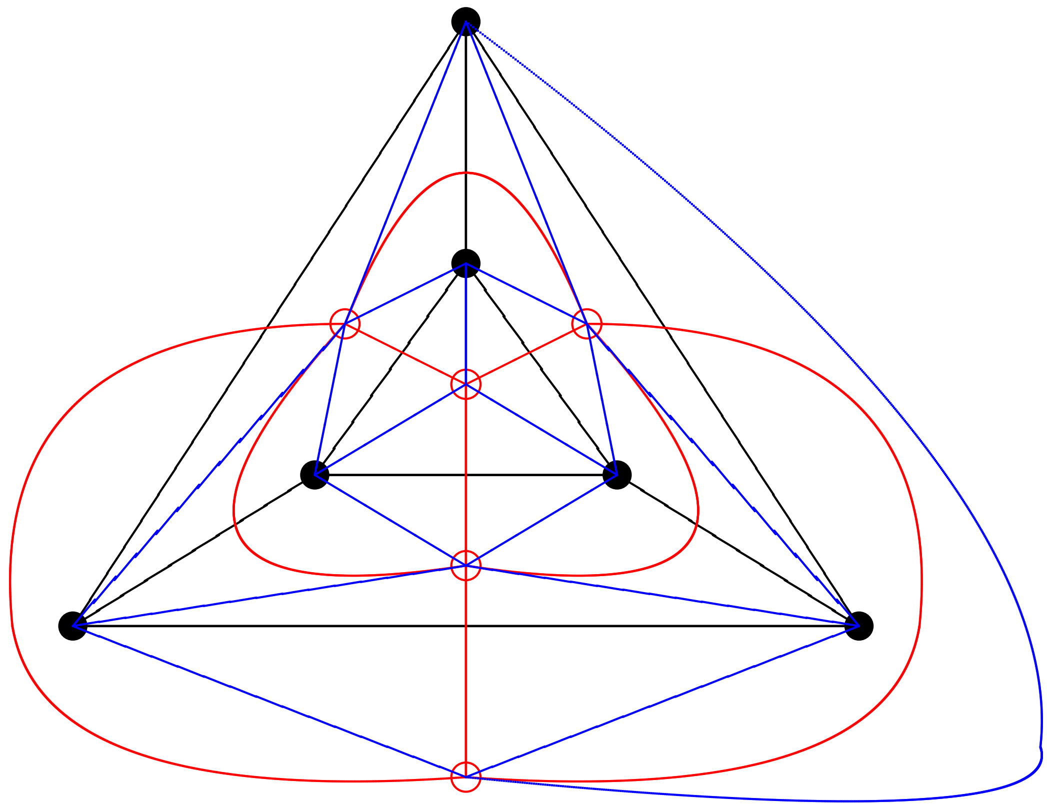
$K_3 \square K_2$

$(K_3 \square K_2)_D$



$K_3 \square K_2$

$(K_3 \square K_2)_D$



$(K_3 \square K_2)_{vf}$

Def A graph G is *k-embeddable* on a surface S_g, a surface of Euler genus g, if G can be drawn on S_g so that no edge crosses more than k other edges.

Def A graph G is *k-embeddable* on a surface S_g, a surface of Euler genus g, if G can be drawn on S_g so that no edge crosses more than k other edges.

Th [R, AM06] If G is 1-embeddable on S_g, then $\chi_\ell(G) \leq \lfloor \frac{9+\sqrt{32g+17}}{2} \rfloor = R(g) \Rightarrow \mu(G) \geq \frac{1}{R(g)}$.

Def A graph G is *k-embeddable* on a surface S_g, a surface of Euler genus g, if G can be drawn on S_g so that no edge crosses more than k other edges.

Th [R, AM06] If G is 1-embeddable on S_g, then $\chi_\ell(G) \leq \lfloor \frac{9+\sqrt{32g+17}}{2} \rfloor = R(g) \Rightarrow \mu(G) \geq \frac{1}{R(g)}$.

Th [AM06] If G is 1-embeddable on S_g and $w(G) \geq 104g - 204$, then $\chi_\ell(G) \leq 8 \Rightarrow \mu(G) \geq \frac{1}{8}$.

Def A graph G is *k-embeddable* on a surface S_g, a surface of Euler genus g, if G can be drawn on S_g so that no edge crosses more than k other edges.

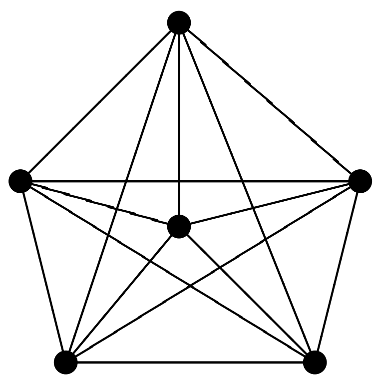
Th [R, AM06] If G is 1-embeddable on S_g, then $\chi_\ell(G) \leq \lfloor \frac{9+\sqrt{32g+17}}{2} \rfloor = R(g) \Rightarrow \mu(G) \geq \frac{1}{R(g)}$.

Th [AM06] If G is 1-embeddable on S_g and $w(G) \geq 104g - 204$, then $\chi_\ell(G) \leq 8 \Rightarrow \mu(G) \geq \frac{1}{8}$.

Q If G is 1-embedded on S_g and $w(G)$ is large enough, is $\mu(G) \geq \frac{1}{6}$?

An Alternate Definition of Q-P

Def [AAPPS95] A graph is *k-quasi*planar* if it has a drawing in which no k of its edges are pairwise crossing.



A 3-quasi*planar graph

Th [AAPPS95] If G is 3-quasi*planar, then $E = \mathrm{O}(n)$.

Th [AAPPS95] If G is 3-quasi*planar, then $E = \mathrm{O}(n)$.

Th [AcT05] If G is simple and 3-quasi*planar, then $E < 6.5n$ The bound is sharp except for a subtractive constant.

Th [AAPPS95] If G is 3-quasi*planar, then $E = O(n)$.

Th [AcT05] If G is simple and 3-quasi*planar, then $E < 6.5n$ The bound is sharp except for a subtractive constant.

Cor If G is 3-quasi*planar then $\mu(G) \geq \frac{1}{13}$.

Th [AAPPS95] If G is 3-quasi*planar, then $E = \mathrm{O}(n)$.

Th [AcT05] If G is simple and 3-quasi*planar, then $E < 6.5n$ The bound is sharp except for a subtractive constant.

Cor If G is 3-quasi*planar then $\mu(G) \geq \frac{1}{13}$.

The graphs which show that the edge bound is sharp have $\mu \geq \frac{1}{6}$.

Th [Ac05] If G is 4-quasi*planar, then $E \leq 36(n-2)$.

Q Do k-quasi*planar graphs have a linear number of edges?

Q If G is k-quasi*planar (especially when $k = 3$), what is the best bound for $\mu(G)$?