Modern Reinforcement Learning Theory

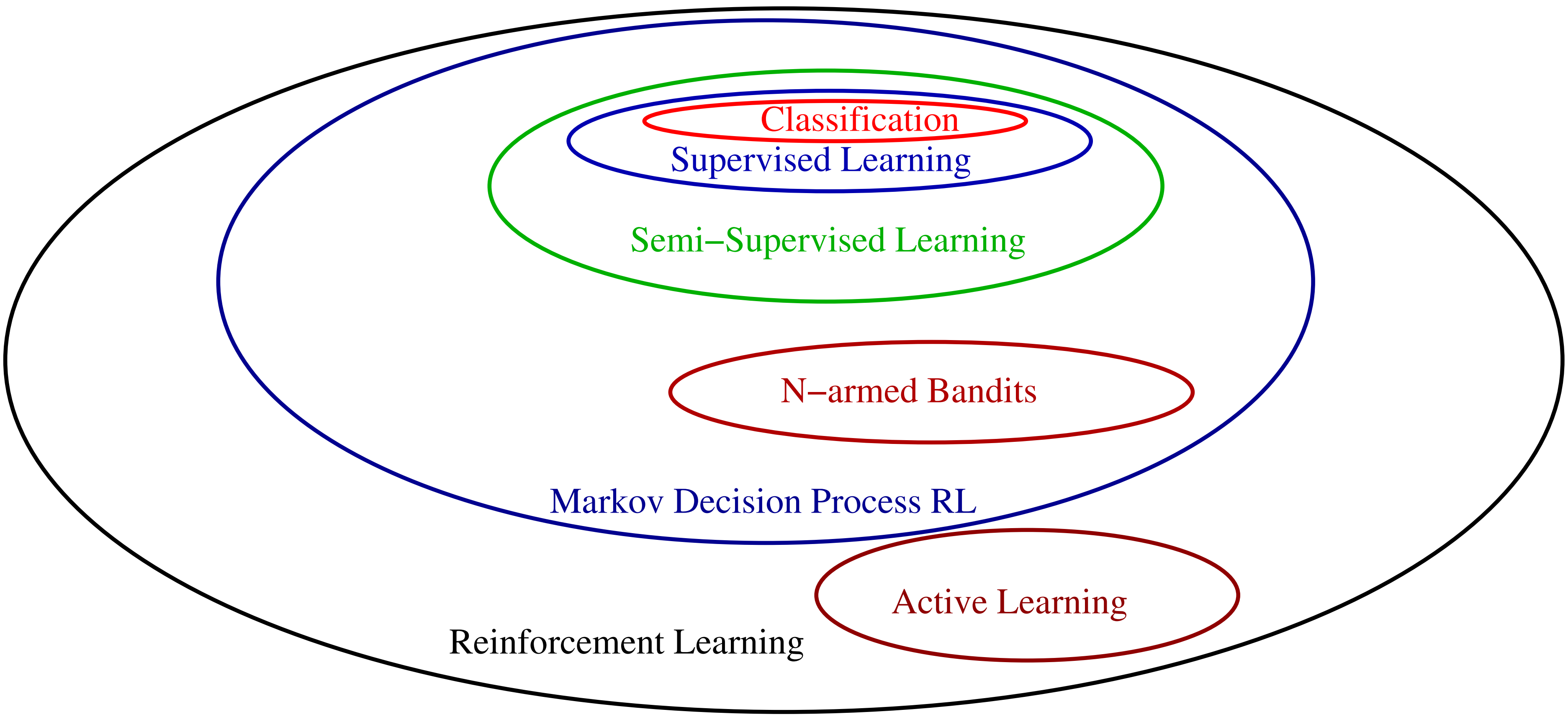
John Langford (with help from many others...)

Yahoo Research

Backing Material: http://hunch.net/~jl/tutorial/RL.html

MLSS 2006, Taiwan, July 25

Reinforcement Learning is Always Relevant



The answer to: "Is this an RL problem?" is always "yes".

The implication: RL theory is broadly applicable.

The other implication: RL theory is often only weakly relevant. (breadth+relevance=hard.)

Understanding a problem as an RL problem is the *beginning* to solving it. Whenever possible, you want to understand how the problem is special.

Outline

1. Sample Complexity Results

2. Limitations of Sample Complexity

3. Reductions Results

What is a sample complexity guarantee?

"With high probability, given $m(\cdot)$ samples, guarantee ϕ holds."

1. S = the number of states in an MDP
2. A = the number of actions/state in an MDP
3. T = the horizon time you care about (or γ = discount factor)
4. O = number of observations
5. ϵ = precision parameter

Important Derived Quantities

$V_t^\pi(s) = E_{(s,a,r)^t \sim \mathrm{MDP}_{s,\pi}} \sum_{t'=1}^{t} r_{t'}$ = the value of being in state s and acting according to π for t timesteps.

$Q_t^\pi(s,a) = E_{(s,a,r)^t \sim \mathrm{MDP}_{sa,\pi}} \sum_{t'=1}^{t} r_{t'}$ = the value of being in state s, acting with a, and then acting according to π for t timesteps.

$\pi^*(s) = \arg\max_a Q_t^{\pi^*}(s,a)$ = recursive definition of optimal policy.

$Q_t^*(s,a) = Q_t^{\pi^*}(s,a)$ = short hand for optimal policy Q values.

The E^3 Guarantee

Trace Model = ability to read current state s, take action a, observe next state s' and reward r. Notation: $TM : A \rightarrow S \times [0,1]$.

Assume MDP$(S, A, p(s'|s,a))$ with horizon T

1. Original: +assume mixing time $\tau \Rightarrow$ Poly$(S, A, \tau, \frac{1}{\epsilon})$ samples implies ability to act ϵ optimal for $T > \tau$.

2. Modified: Poly$(S, A, T, \frac{1}{\epsilon})$ samples implies ability to act ϵ optimal for T timesteps.

(2) + mixing assumption implies (1). (2) holds even for deterministic worlds. We'll go through (2).

E^3 Theorem Statement

Theorem: There exist an algorithm E^3 such that for all MDP $(S, A, T, p(s'|a, s))$ with rewards $r \in [0, 1]$, with probability $1 - \delta$, for all except Poly$\left(S, A, T, \frac{1}{\epsilon}, \ln\frac{1}{\delta}\right)$ steps $Q^{E^3}_{T-t \text{ mod } T}(s, E^3(h)) \geq V^*_{T-t \text{ mod } T}(s) - \epsilon$ where h is the history of observations.

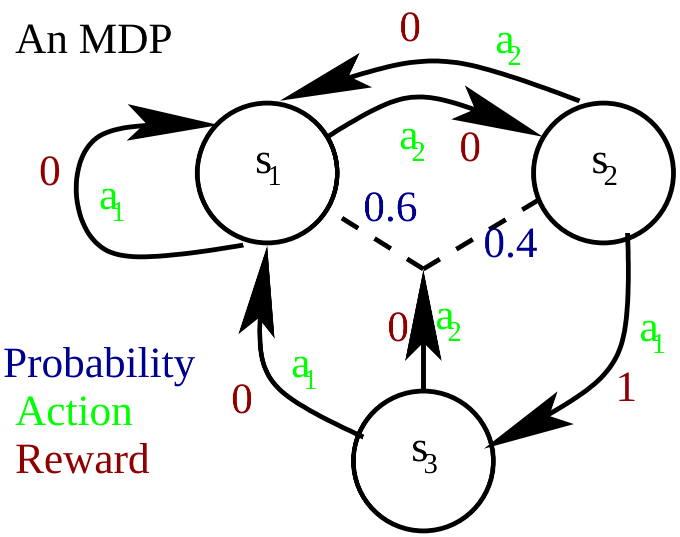
Suboutline:

1. The Algorithm

2. The Proof

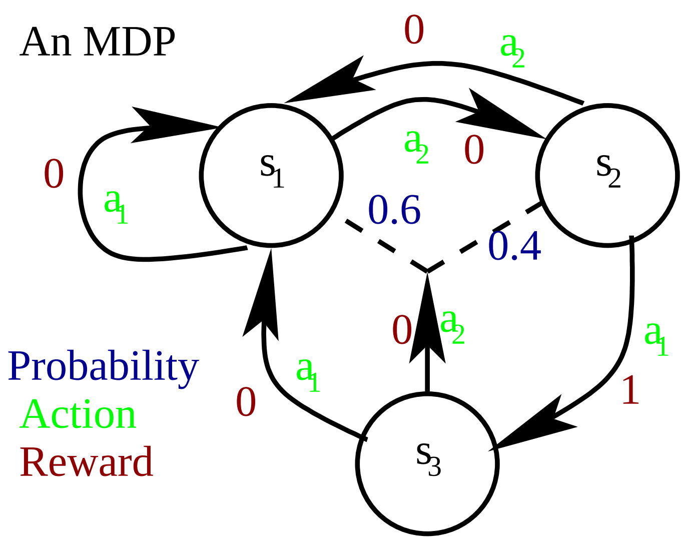
The Known(h) MDP

A state s, all actions a leaving s and the probability of their outcomes is known if all actions a leaving s have been executed at least n times.

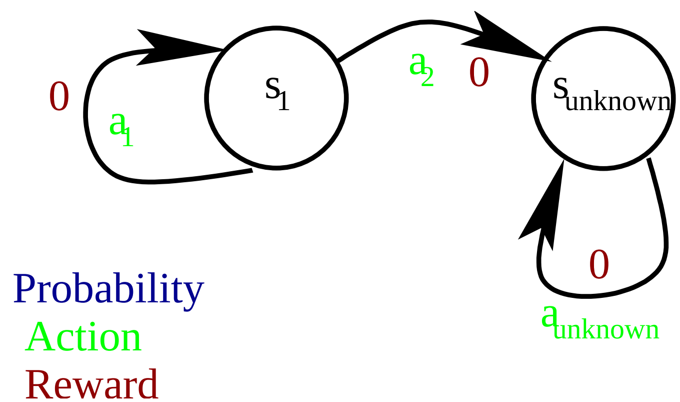


Initially: known MDP $=$ nothing

The Known(h) MDP

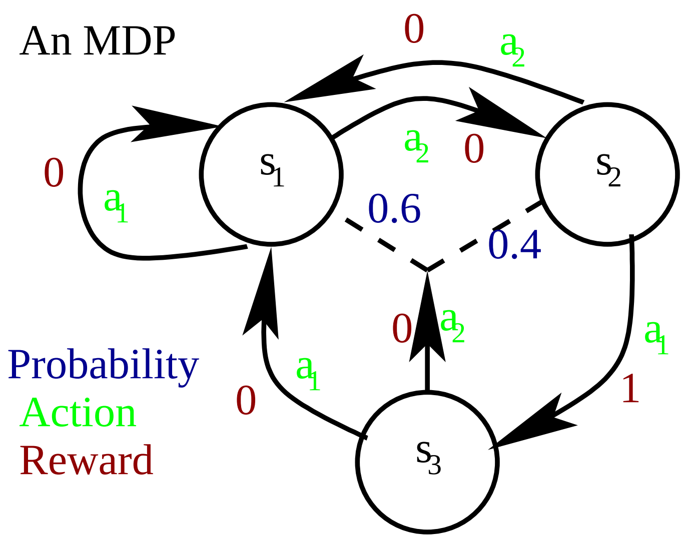


Then:

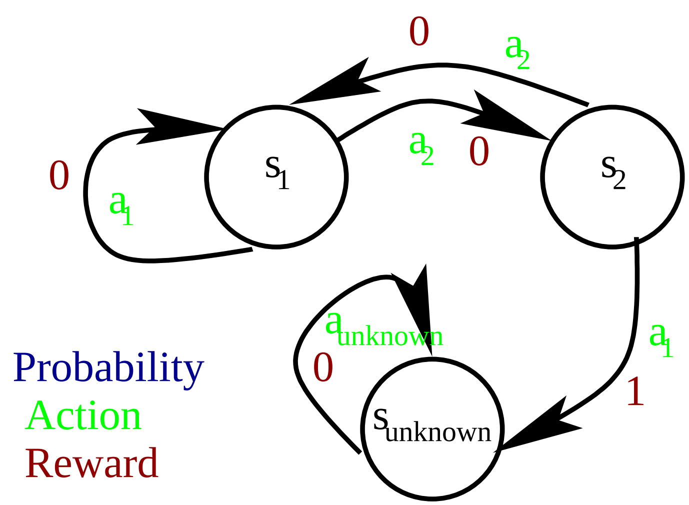


Complete dangling action(s) with one state that always has reward 0.

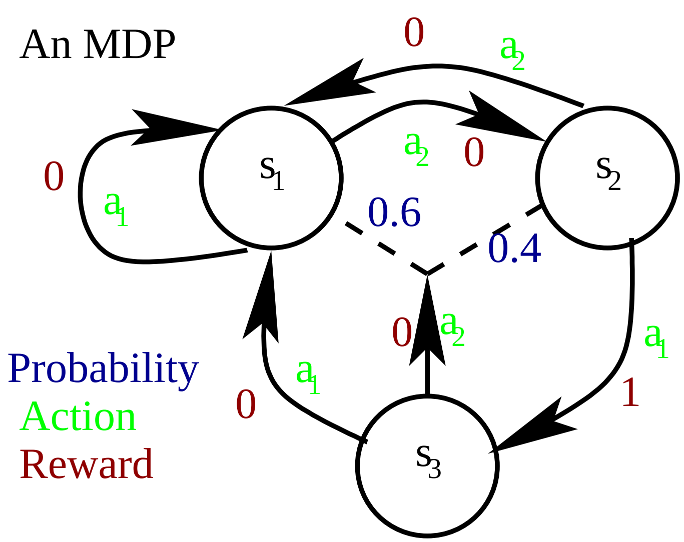
The Known(h) MDP



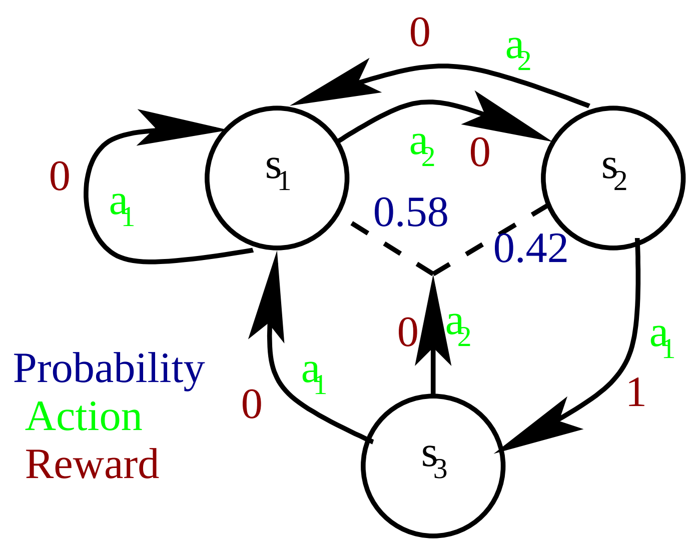
Then:



The Known(h) MDP



Finally:



(note: the probabilities are empirical counts)

The Unknown(h) MDP

Unknown(h) = Known(h) except the reward is 1 for actions which leave the known states and 0 otherwise.

Dynamic Program

Fundamental operation: Given MDP M and state s,

$$\mathsf{DP}(M, s, t) = a, v$$

where v = the maximum expected $T - (t \text{ mod } T)$ reward sum and a= action achieving it.

Computation:

$$\mathsf{DP}(M, s, t) = \max_a E_{s',r \sim M(s,a)} r + \mathsf{DP}(M, s', t+1)$$

$$\mathsf{DP}(M, s, nT) = 0$$

$E^3(h)$ Explicit Explore or Exploit Algorithm

1. If last s not in Known(h): choose the least previously used action

2. Else:

   (a) If DP(Unknown(h)) $> \epsilon'$ then act according DP(Unknown(h)) until state is unknown or $t \bmod T = 0$ then go to (1).

   (b) else act according to DP(Known(h)).

The proof uses 5(!) MDPs

1. MDP — the true MDP (Imposed by world)

2. Known(h) = known MDP (Known by E^3 algorithm)

3. Unknown(h) = unknown MDP (Known by E^3 algorithm)

4. $\text{MDP}_{\text{K}(h)}$ = MDP restricted to the known states (exists only in proof)

5. $\text{MDP}_{\text{U}(h)}$ = MDP restricted to the known states with rewards set to 0 except for escaping rewards. (exists only in proof)

Proof Sketch:

Simulation Lemma:

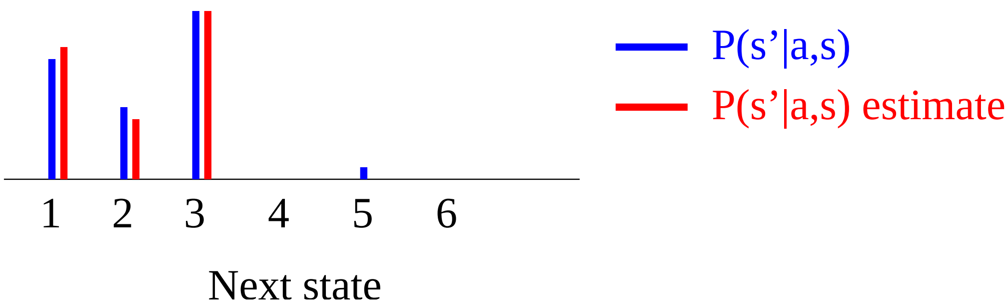
$$|\mathsf{DP}(\mathsf{MDP}_{\mathsf{U/K}(h)}) - \mathsf{DP}((\mathsf{Un})\mathsf{Known}(h))| \leq \frac{1}{\mathsf{Poly}(S, A, T, \ln \frac{1}{\delta})}$$

Explore/Exploit Lemma:

$$\mathsf{DP}(\mathsf{MDP}_{\mathsf{K}(h)}) + T\mathsf{DP}(\mathsf{MDP}_{\mathsf{U}(h)}) \geq \mathsf{DP}(\mathsf{MDP})$$

So $n = \mathsf{Poly}(S, A, T, \ln \frac{1}{\delta})$ implies ability to simulate on known states to precision $\frac{1}{\mathsf{Poly}(S,A,T,\ln \frac{1}{\delta})} << \epsilon$. $\Rightarrow$ Explore/Exploit Lemma implies $\mathsf{DP}(\mathsf{MDP}) - \mathsf{DP}(\mathsf{MDP}_{\mathsf{K}(h)}) > \epsilon \Rightarrow \mathsf{DP}(\mathsf{MDP}_{\mathsf{U}(h)}) > \frac{\epsilon}{T}$ $\Rightarrow$probability about $\frac{\epsilon}{T}$ of encountering new state if exploring. This can happen only $O(\frac{nSAT}{\epsilon})$ times (Using the Chernoff bound). Each exploration uses at most T steps $\Rightarrow$ proof.

Simulation Lemma Proof sketch



Enough samples $\Rightarrow$ small l_1 distance $d = \sum_{s'} |\widehat{p}(s'|a,s) - p(s'|a,s)|$.

l_1 distance d $\Rightarrow$ with probability $1 - \frac{d}{2}$ draw from $p(s'|a,s)$.

$\Rightarrow 1 - T\frac{d}{2}$ draw from T-length sequence.

$\Rightarrow$ Expectations correct up to $T\frac{d}{2}$.

So how do we prove l_1 convergence?

Each next state s' = IID draw.

n IID coin flips converge to mean like $O(\frac{1}{\sqrt{n}})$ = Chernoff bound.

$$\Rightarrow \text{Pr}\left(|\hat{p}(s'|a,s) - p(s'|a,s)| \leq \sqrt{\frac{\ln\frac{2}{\delta}}{2n}}\right) \geq 1-\delta$$

Union $\Rightarrow \text{Pr}\left(\forall s', s, a:\ |\hat{p}(s'|a,s) - p(s'|a,s)| \leq \sqrt{\frac{\ln\frac{2S^2A}{\delta}}{2n}}\right) \geq 1-\delta$

Only S next states s' so $1-\gamma$ probability on $\{s' : p(s'|a,s) > \frac{\gamma}{S}\}$ so $n = \tilde{O}\left(\frac{S^2}{\gamma^2}\right)$ implies a $1-\gamma$ fraction of $\hat{p}(s'|a,s)$ indistinguishable from $p(s'|a,s)$.

Explore or Exploit Lemma Proof Sketch

$\text{DP}(\text{MDP}) = E_{(s,a,r)^T \sim \pi^*, \text{MDP}} \left[\sum_t r_t\right]$

Let $v = (s,a,r)^T \in \text{Known}(h)$

Let $p = \Pr_{\pi^*, \text{MDP}}(v)$

$$= pE_{(s,a,r)^T \sim \pi^*, \text{MDP}|v} \left[\sum_t r_t\right] + (1-p)E_{(s,a,r)^T \sim \pi^*, \text{MDP}|\bar{v}} \left[\sum_t r_t\right]$$

$$\leq pE_{(s,a,r)^T \sim \pi^*_{\text{K}(h)}, \text{MDP}_{\text{K}(h)}} \left[\sum_t r_t\right] + \text{DP}(\text{MDP}_{\text{U}(h)}) E_{(s,a,r)^T \sim \pi^*, \text{MDP}|\bar{v}} \left[\sum_t r_t\right]$$

$$\leq \text{DP}(\text{MDP}_{\text{K}(h)}) + \text{DP}(\text{MDP}_{\text{U}(h)})T$$

R-Max(h) Modification

(Roughly) The same result holds for the following algorithm:

1. If last s not in Known(h): choose the least previously used action.

2. Else act according to DP(Known(h)+Unknown(h)).

(Where "+" adds the rewards of each MDP on each edge.)

Delayed Q-learning

The theorem can be tightened from $\text{Poly}(S, A)$ to $\tilde{O}(SA)$ using the Delayed Q-learning algorithm.

Outline

1. Sample Complexity Results

2. Limitations of Sample Complexity

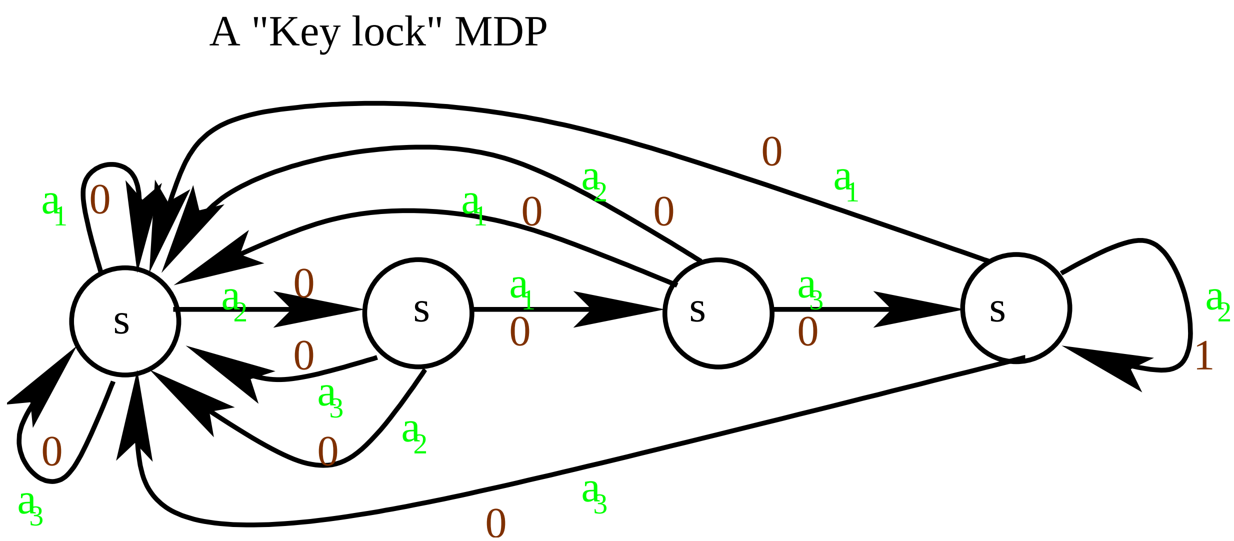
3. Reductions Results

The Limits of Sample Complexity: A lower bound

Theorem: Any algorithm A satisfying the E^3 statement must use at least $\Omega(TSA)$ actions to explore.

(There are stronger lower bounds, but this is sufficient.)

Proof



States in a chain. One action leads to next state, all the rest lead to the beginning. The final state has an action with reward 1.

Implications

Lower bound $\Rightarrow$ the really big problems can't be solved.

But the problems *are* solvable: we solve them every day.

$\Rightarrow$More or different assumptions are required.

Attempt 1: Factored-E^3

Assume state is represented by a set of bits $s = (b_1, b_2, ..., b_n)$.

Assume $P(s'|s,a) = \Pi_i P(b_i|B_i,a)$ where $B_i =$ subset of $\{b_1, b_2, ..., b_n\}$.

Theorem: There exist an algorithm Factored-E^3 such that for all Factored MDP, with probability $1-\delta$, for all except

$$\text{Poly}\left(|MDP|, T, \frac{1}{\epsilon}, \ln\frac{1}{\delta}\right)$$

steps $Q^{\text{Factored}-E^3}_{T-t \bmod T}(s, \text{Factored} - E^3(h)) \geq V^*_{T-t \bmod T}(s) - \epsilon$ where h is the history of observations and $|MDP|$ = description length.

Problem 1: Factoring isn't enough. Consider this room...

Problem 2: The *computational* complexity of planning is high.

Attempt 2: Metric-E^3

Assume there exists Model$(\{(s,a,r,s')^*\},s'')$ such that with m experiences satisfying $d((s,a),(s'',a'')) \le \alpha$, Model outputs s',r from almost the same distribution as $P(s',r|s'',a'')$.

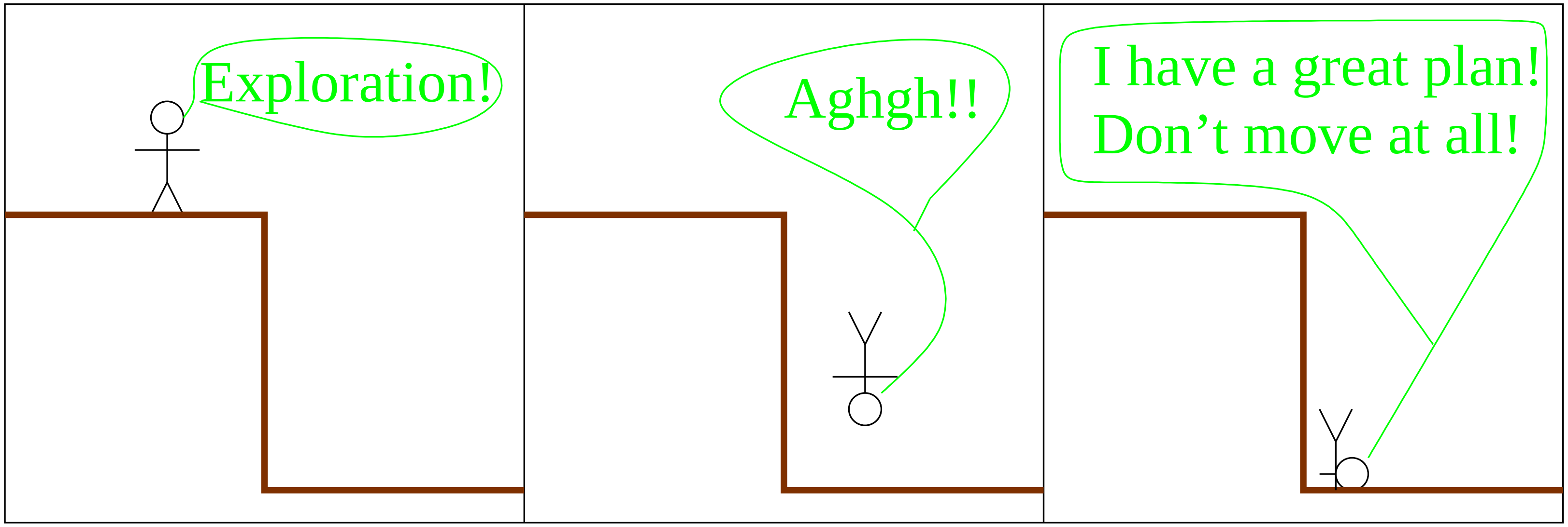
Theorem: There exist an algorithm Metric-$E^3(h)$ such that for all Metric MDPs, with probability $1-\delta$, for all except

$$\text{Poly}\left(Cover(MDP), T, \frac{1}{\epsilon}, \ln\frac{1}{\delta}\right)$$

steps $Q^{\text{Metric-}E^3}_{T-t \bmod T}(s, E^3(h)) \ge V^*_{T-t \bmod T}(s) - \epsilon$ where h is the history of observations.

Problems: Can we reasonably expect Model to be that accurate?

Do we really want the guarantee these algorithms provide?



See Abbeel & Ng ICML 2005 for a bit of discussion.

Outline

1. Sample Complexity Results
2. Limitations of Sample Complexity
3. Reductions Results

Reduction-style

Philosophy: Start with what we know we can do (even if we can't prove it), then use it repeatedly to see what we can do.

In Machine Learning, we know how to classify. Can we (re)use this ability to solve Reinforcement Learning?

General form: "cost sensitive" classification D on $X \times [0, \infty)^k$:

$$\arg\min_h E_{x,\vec{c} \sim D} \left[c_{h(x)} \right]$$

(Reducible to binary classification. Reduction suggests minimizing $\sum_i c_i$ is good.)

How do we think about a general RL problem?

$D(o', r|(o, a, r)^*, o)$ = conditional probability table

1. $o', o \in O$ = observations

2. $r \in [0, \infty)$ = reward

3. $a \in A$ = action:

Find $\pi((o, a, r)^*, o) \rightarrow a$ maximizing:

$$\eta(\pi) = E_{(o,a,r)^T \sim \pi, D}\left[\sum_{t=1}^{T} r_t\right]$$

An "Algorithm"

Let π^*= optimal policy

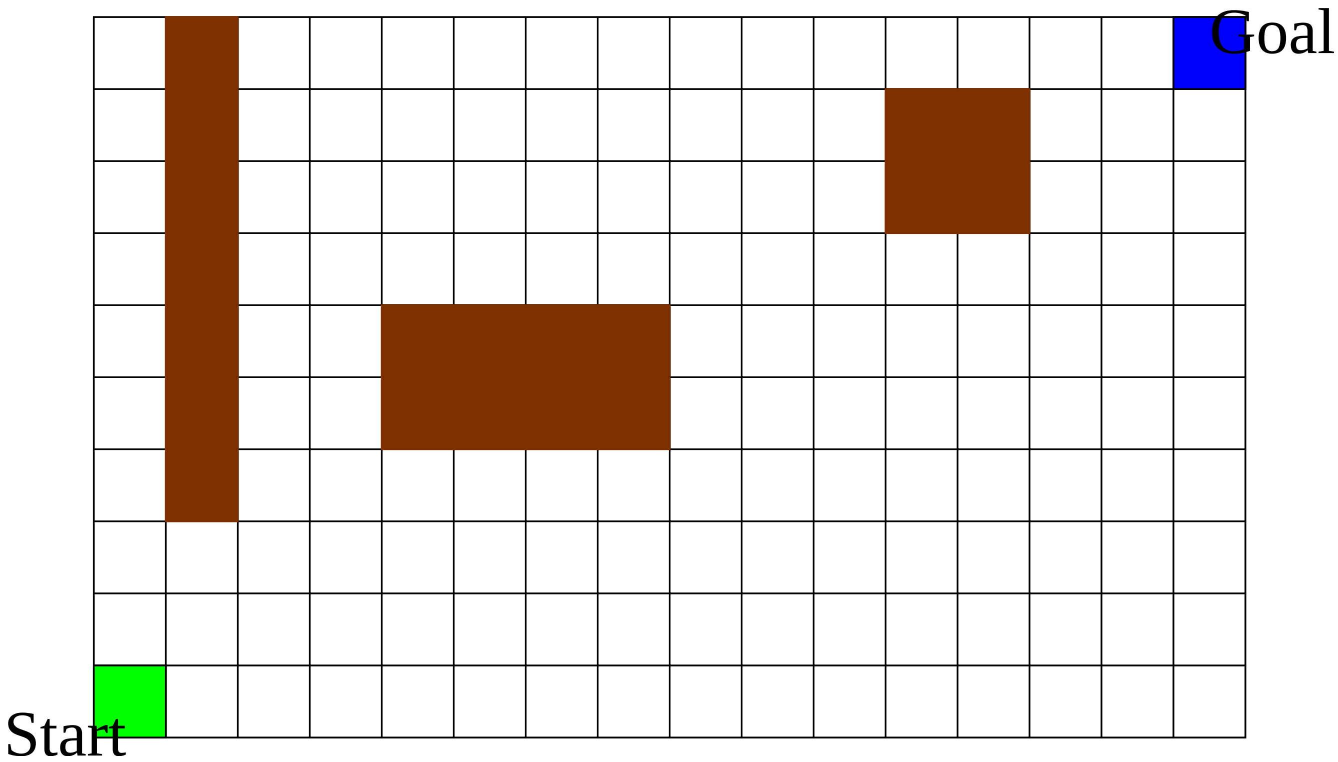
1. Given empty history, predict optimal a assuming π^* followed for $T-1$ timesteps afterwards.

2. Given first prediction, 1 step history, predict optimal a assuming π^* followed for $T-2$ timesteps afterwards.

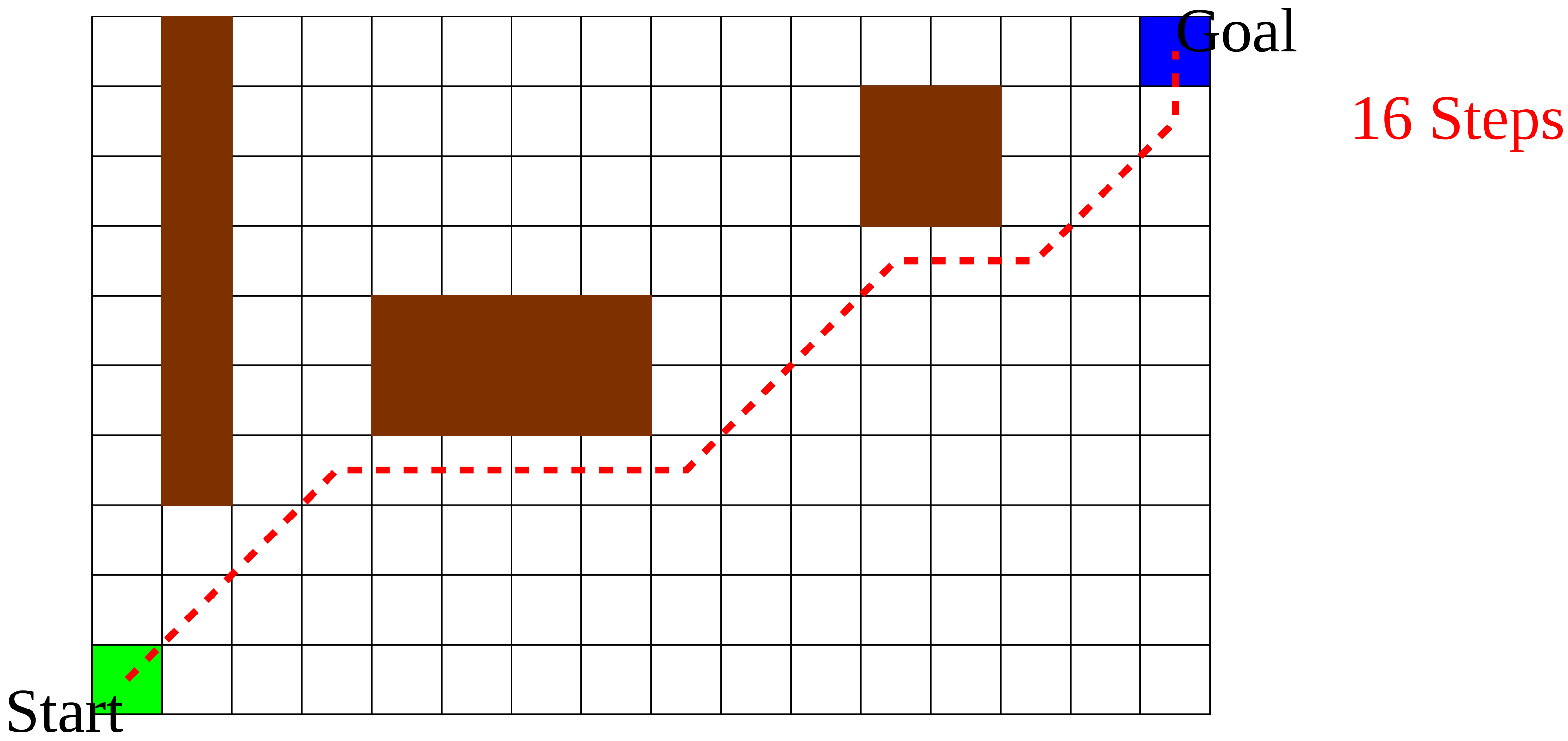
3. Given 2 step history, predict optimal a assuming π^* followed for $T-3$ timesteps afterwards.

4. ...

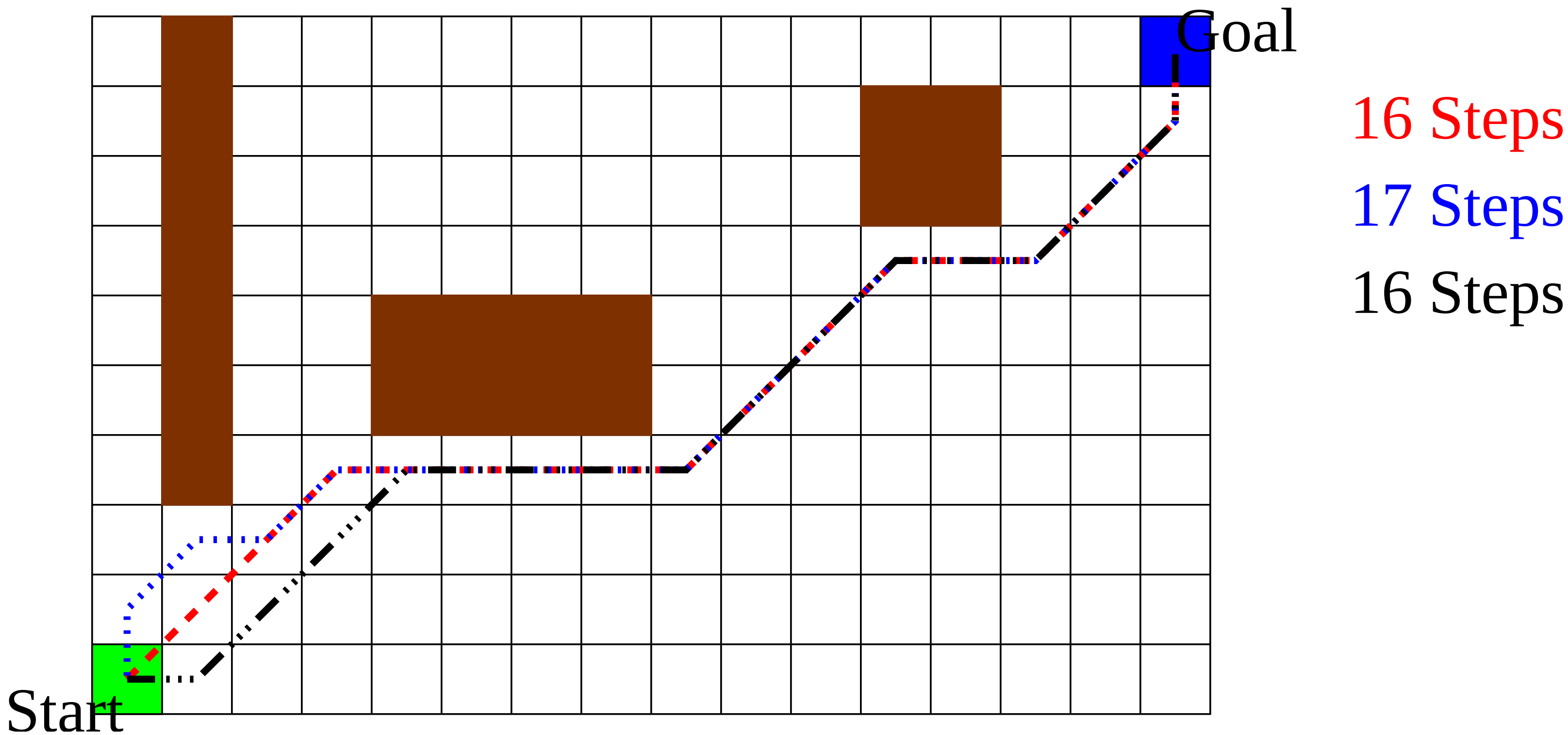
The Algorithm: Pretty Pictures



The Algorithm: Pretty Pictures

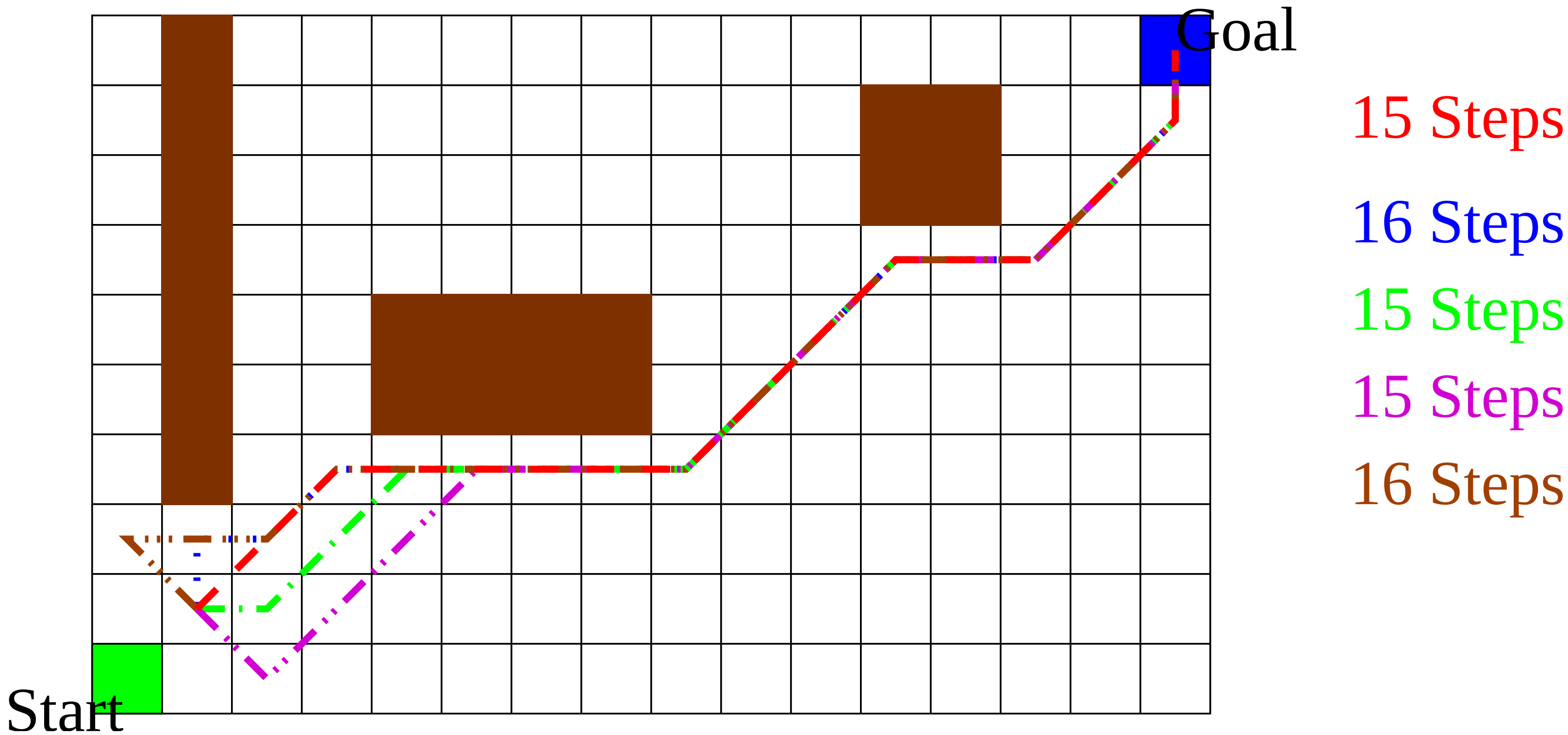


The Algorithm: Pretty Pictures



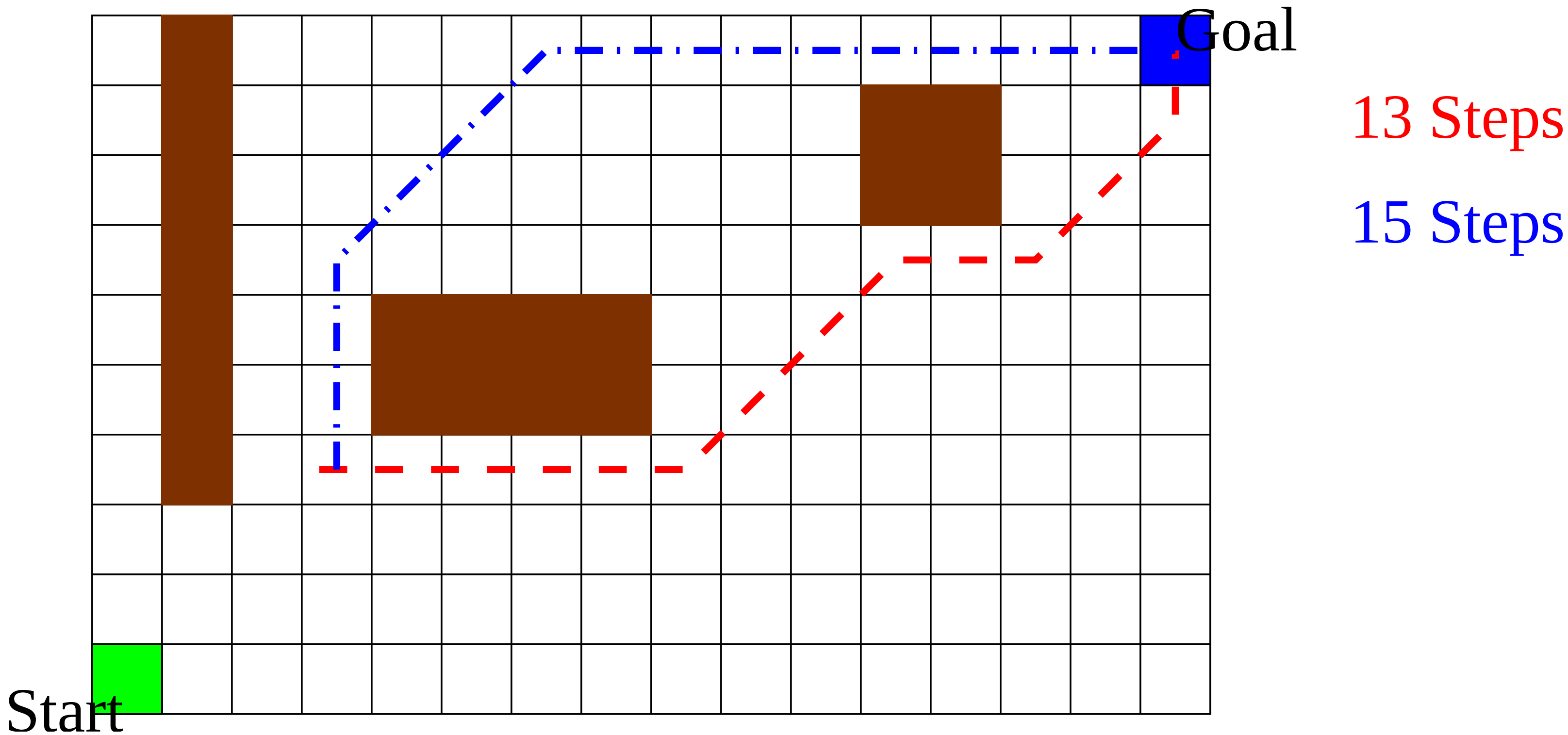
Cost for action $a = \text{steps}(a) - \min_{a'} \text{steps}(a') = 0,\ 1,\ 0$

The Algorithm: Pretty Pictures



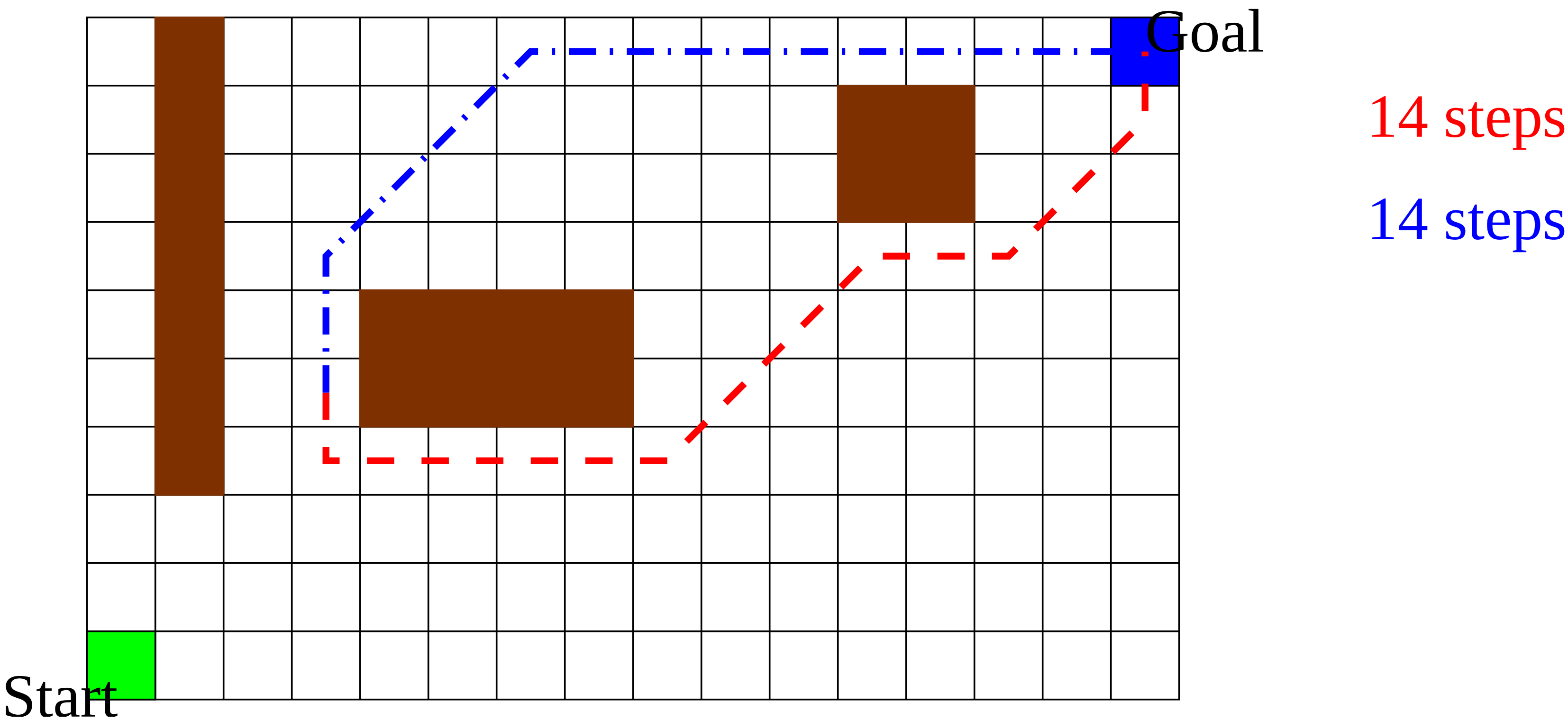
cost = 0, 1, 0, 0, 1

The Algorithm: Pretty Pictures



Cost = 0, 2

The Algorithm: Pretty Pictures



Cost = 0, 0 Note: condition on suboptimal outcome at previous step.

The Critical Lemma

$A^*_t(D,\pi,h_t) = E_{(a,o,r)^T\ D,(\pi,h_t,\pi^*)}\left[\sum_{t'=t}^{T} r_{t'}\right]$ = Expected reward sum when starting with π, acting according to h_t and completing with the optimal policy. (A for "Advantage".)

Lemma: For all D, $\pi = (h_1, ..., h_T)$,

$$\eta(\pi^*) - \eta(\pi) = \sum_{t=1}^{T} A^*_t(D,\pi,\pi^*_t) - A^*_t(D,\pi,h_t)$$

= sum of cost sensitive losses.

Proof

$$\eta(\pi^*) - \eta(\pi) = E_{(o,a,r)^T \sim D,\pi^*}\left[\sum_{t=1}^{T} r_t\right] - E_{(o,a,r)^T \sim D,\pi}\left[\sum_{t=1}^{T} r_t\right]$$

$$= \sum_{t'=1}^{T} E_{(o,a,r)^T \sim D,(\pi,\pi^*_{t'},\pi^*)}\left[\sum_{t=1}^{T} r_t\right] - E_{(o,a,r)^T \sim D,(\pi,\pi_{t'},\pi^*)}\left[\sum_{t=1}^{T} r_t\right]$$

$$= \sum_{t'=1}^{T} E_{(o,a,r)^T \sim D,(\pi,\pi^*_{t'},\pi^*)}\left[\sum_{t=t'}^{T} r_t\right] - E_{(o,a,r)^T \sim D,(\pi,h_{t'},\pi^*)}\left[\sum_{t=t'}^{T} r_t\right]$$

$$= \sum_{t=1}^{T} A^*_t(D,\pi,\pi^*_t) - A^*_t(D,\pi,h_t)$$

Ways to think about the result

1. Reduction of RL to classification given oracle access to optimal policy. (A form of “apprenticeship learning”.) [Surprisingly useful.]

2. A *nonconstructive* reduction of RL to classification. There exists a set of classification problems such that if they are solved well, we solve RL. (But which problem is unclear.)

What can we do constructively?

Given side information, we can subvert the exploration problem.

1. Generative model (GM) (high exponential)

2. Training time Acces to Optimal Policy + GM

3. State Distribution of (near) optimal policy + GM

Sparse Sampling

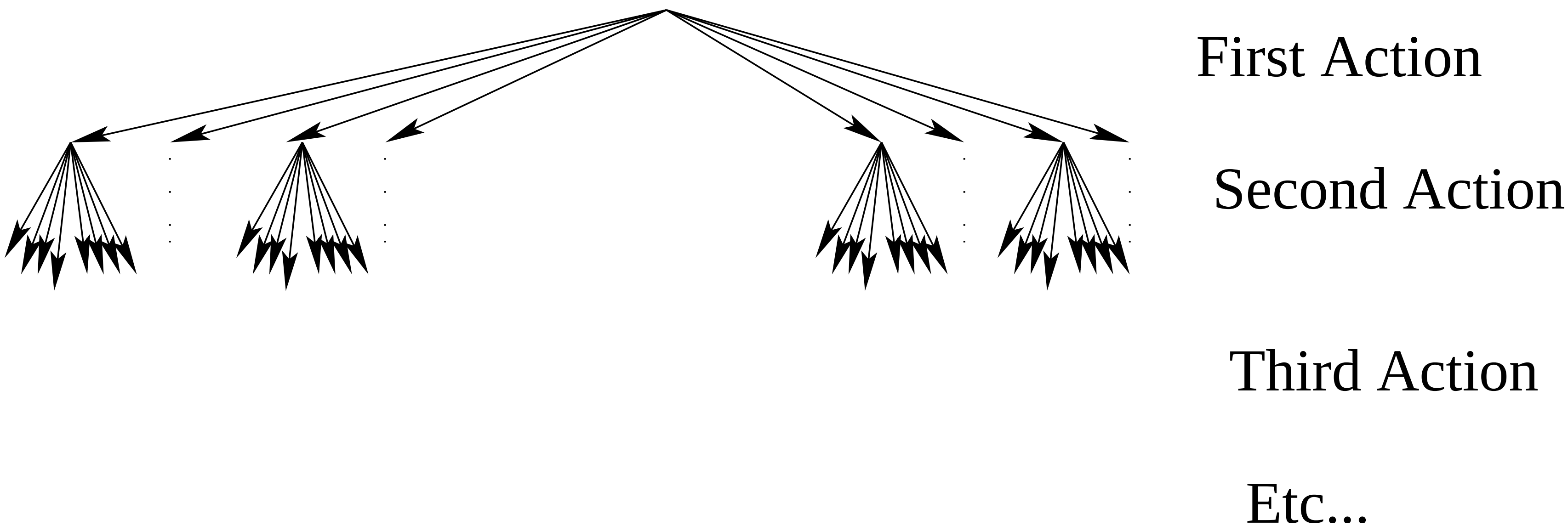
1. For every first action, sample k times

2. For every result, for every second action, sample k times.

3. For every result, for every third action, sample k times.

4. etc..

Sparse Sampling, the picture



Picture is difficult, very high sample complexity.

Sparse Sampling Decisions

1. For each action, expectation over samples at step T
2. Max over actions.
3. For each action, expectation over T maxes at step $T-1$.
4. Etc...

$\Rightarrow$ Estimate of $l_t(D,\pi,h_t)$ accurate to $(T-t)\sqrt{\frac{\log A+\log\frac{1}{\delta}}{k}}$.

$\Rightarrow$ can apply reduction

What can we do constructively?

Given side information, we can subvert the exploration problem.

1. Generative model (GM) (high exponential)
2. Training time Acces to Optimal Policy + GM
3. State Distribution of (near) optimal policy + GM

Training time Acces to Optimal Policy + GM

Algorithm from before applies.

But: Learning new classifier for every timestep is computationally difficult.

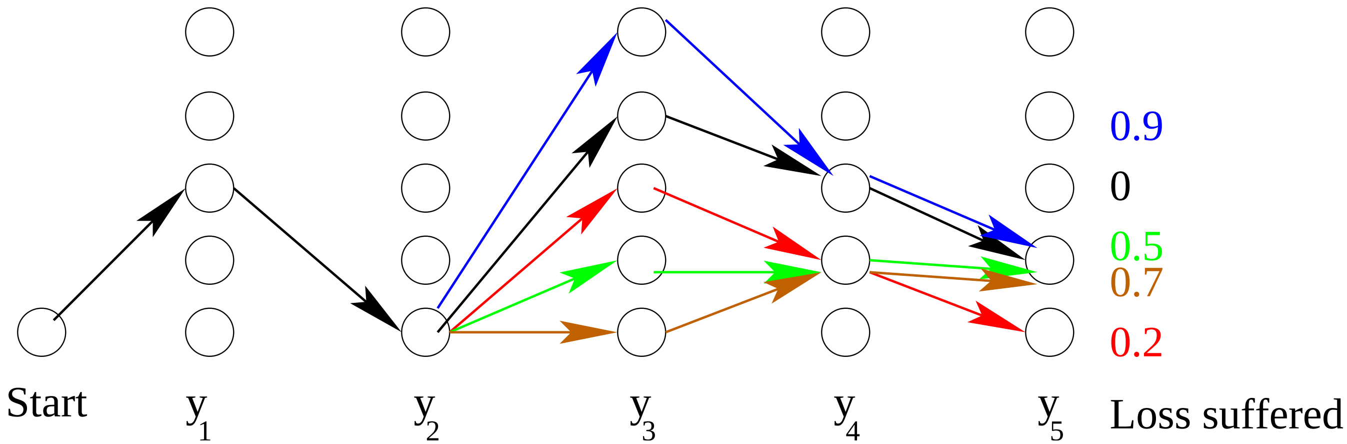
Solution: Learn one classifier for all timesteps simultaneously.

But: This breaks good reduction properties.

Solution: Use “Conservative Policy Iteration” trick.

The Conservative Policy Iteration trick

We have policy π_{i-1} which we use to define examples for h using GM:



Iteration = $\pi_i \leftarrow (1-\beta)\pi_{i-1} + \beta h$ (stochastic interpolation)

If you start with π^*, after $O\left(\frac{1}{\beta}\right)$ iterations, the policy is entirely learned.

The Searn Theorem

Theorem: For *all* RL problems D, for *all* sequences of learned classifiers $c_1, c_2, ...$, for small β, after $2T^3 \ln T$ iterations,

$$\text{Loss(learned policy)}$$

$$\leq \text{Loss(Initial Policy)} + \text{Average classifier loss} * 2T \ln T + (1 + \ln T)\frac{\text{max loss}}{T}$$

Note: This is actually *worse* than $O(T)$ steps. However, in practice, line search over β implies $< T$ classifiers learned.

Searn Experimental Results

Spanish Named Entity Recognition (1517 test)

	CRF(300)	SVM(300)	SVMISO(300)	Searn(300)	Searn(8324)
Acc.	94.83	94.94	94.9	95.01	97.67

(Multidigit) Handwriting Recognition (5500 or 600 test)

Training set size	Perceptron	Logistic	SVM	M^3N	Searn
600 train	65.56	68.65	82.63	87.5	88.2
5500 train	70.05	72.10	82.52	?	90.91

Joint sequence labeling (8936 train, 2012 test)

	Inc. P.	LASO	Factored CRF	CRF	Searn
Joint Accuracy	93.12	95.12	96.48	?	96.81
Chunk F-score	92.44	93.49	93.87	94.77	94.36
Parts of speech	98.05	98.90	98.92	?	99.07

Document Summarization (fraction of max "Rouge-1+2")

word count	DUC 05 winner	BD03	BD05	Searn	Searn (word)
100	?	0.803	0.900	0.971	1.129
250	0.749	0.696	0.883	0.906	1.170

What can we do constructively?

Given side information, we can subvert the exploration problem.

1. Generative model (GM) (high exponential)

2. Training time Acces to Optimal Policy + GM

3. State Distribution of (near) optimal policy + GM

State Distribution of (near) optimal policy + GM

What if (near) optimal policy is missing?

Sometimes we only know where an optimal policy visits. Is this enough?

The PSDP ("Policy Search by Dynamic Programming") algorithm

Let D_1, ..., D_T = distribution over states at timestep t induced by π^*.

1. Repeatedly draw s from D_T and use GM to estimate rewards. Learn a Tth step policy π_T via cost sensitive classification.

2. Repeatedly draw s from D_{T-1} and use GM + π_T to estimate rewards. Learn $(T-1)$th step policy π_{T-1} via cost sensitive classification.

3. etc...

The PSDP Theorem

$A_t^\pi(D_t, h_t) = E_{(a,o,r)^T \ D_t,(h_t,\pi)}\left[\sum_{t'=t}^T r_{t'}\right]$

(Like A_t^* except we *start* by assuming optimal policy and *end* with learned policy.)

Theorem: For all π^* generating $D_1, ..., D_T$, and all $\pi = (h_1, ..., h_T)$,

$$\eta(\pi^*) - \eta(\pi) = \sum_{t=1}^{T} A_t(D_t, \pi_t^*) - A_t(D_t, h_t)$$

= sum of cost sensitive losses/rewards.

- The same as given $\pi^* +$ GM but Searn-style variant unknown.

The Future

1. Specializations of Reinforcement Learning to particular domains.
   (a) Robotics
   (b) Games
   (c) Stock Market
2. Exploration: the elephant in the closet.
3. New models and new methods of analysis needed and welcome.

Related Reading

[E^3] Michael Kearns and Satinder Singh, "Near Optimal Reinforcement Learning in Polynomial Time", ICML 1998.

[R-Max] Ronen Brafman and Moshe Tennenholtz, "R-Max - A General Polynomial Time Algorithm for Near Optimal Reinforcement Learning", IJCAI 2001.

[Factored-E^3] Michael Kearns and Daphne Koller, "Efficient Reinforcement Learning in Factored MDPs", IJCAI 1999.

[Metric-E^3] Sham Kakade, Michael Kearns, and John Langford, "Exploration in Metric State Spaces", ICML 2003.

[Delayed Q-learning] Alexander Strehl et al, "PAC Model-Free Reinforcement Learning", ICML 2006.

[Sparse Sampling] Michael Kearns, Yishay Mansour, and Andrew Ng, "A Sparse Sampling Algorithm for Near-Optimal Planning in Large Markov Decision Processes", IJCAI 1999.

[CPI] Sham Kakade, John Langford "Approximately Optimal Approximate Reinforcement Learning", ICML 2002.

[Many things] Sham Kakade, On the Sample Complexity of Reinforcement Learning Thesis Gatsby, UCL, 2003.

[PSDP] Drew Bagnell, Sham Kakade, Andrew Ng, & Jeff Schneider, "Policy Search by Dynamic Programming", NIPS 2003.

[Given optimal Policy] John Langford and Bianca Zadrozny, "Relating Reinforcement Learning Performance to Classification Performance", ICML 2005.

[Searn] Hal Daume III, John Langford, and Daniel Marcu, "Search-Based Structured Prediction" (unpublished).

Extra Discussion on many things, including RL, Reductions, etc...

http://hunch.net