Label optimal regret bounds for online local learning

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Online local learning framework (Christiano ‘14)

Learning Algorithm (A)

observe item pair $(x_t, y_t) \in \mathcal{X} \times \mathcal{X}$

predict label pair $(L(x_t), L(y_t)) \in \mathcal{L} \times \mathcal{L}$

observe payoff $c_t : \mathcal{L} \times \mathcal{L} \rightarrow [0, 1]$

- Minimize regret: $OPT - E[payoff(A)];$ OPT is payoff of best fixed labeling:

$$OPT = \max_{h \in \mathcal{K}} \sum_{t=1}^{T} c_t(h(x_t), h(y_t))$$

$\mathcal{K}$: set of all possible labelings
Prior work

- Optimal algorithm (information theoretically): experts/MW -- regret $O(\sqrt{n \log lT})$
- OCO interpretation of experts:
  \[
  \mu_{t+1} = \arg \max_{\mu \in \mathcal{D}(\mathcal{K})} \left[ \nu \sum_{s=1}^{t} E_{\mu}[\text{payoff}_s] - H(\mu) \right]
  \]
- Relaxation:
  - OGD over 2\textsuperscript{nd} order pseudo-moments polytope.
  - Regularizer: log-det of second-moment matrix.
- (Christiano ’14): Follow the regularized leader with log-det regularizer achieves regret $O(\sqrt{n l^3 T})$
- (COLT ‘14): Close gap?

$n$: number of items
$l$: number of labels

$H(\mu)$ - Shannon entropy
$\mathcal{D}(\mathcal{K})$ - polytope of distributions over labelings.
Our results

- **Theorem 1**: Follow the regularized leader with log-det regularizer achieves regret $O(\sqrt{nlT})$.

- **Theorem 2**: Achieving regret $\sqrt{nl^{1-6\epsilon}T}$ refutes the planted clique conjecture for planted clique of size $n^{\frac{1}{2}-\epsilon}$.

- $G(n, \frac{1}{2})$ – Erdős–Rényi graph.
- $G(n, \frac{1}{2}, k)$ – Erdős–Rényi graph with a randomly planted clique of size $k$.  

**Planted clique conjecture:**

Distinguishing between $G(n, \frac{1}{2})$ and $G(n, \frac{1}{2}, n^{\frac{1}{2}-\epsilon})$ with const. probability is impossible in polynomial time.
Our results

- **Theorem 1**: Follow the regularized leader with log-det regularizer achieves regret.

- **Theorem 2**: Achieving regret refutes the planted clique conjecture for planted clique of size $n^{1/2-\epsilon}$.

  - $G(n, 1/2)$ – Erdős–Rényi graph.
  - $G(n, 1/2, k)$ – Erdős–Rényi graph with a randomly planted clique of size $k$.

  Planted clique conjecture: Distinguishing between $G(n, 1/2)$ and $G(n, 1/2, n^{1/2-\epsilon})$ with constant probability is impossible in polynomial time.

- **Online version of (Raghavendra ‘08) result for CSPs**: For CSPs, under UGC, pseudo-moment relaxation is optimal.

  - Online: same relaxation is optimal with appropriate regularization.
Robust versions of planted clique

- $G(n, p, k, q)$ – Erdős–Rényi graph of density $p$ with a randomly planted Erdős–Rényi subgraph of size $k$, density $q$.

**Theorem 3:** Achieving regret $\sqrt{nl^{1-\Theta(\alpha, \epsilon) T}}$ refutes the planted dense subgraph conjecture.

**Planted dense subgraph conjecture:**
Distinguishing between $G(n, p)$ and $G(n, p, k, q)$ with constant probability is impossible in polynomial time for

$$p = n^{-\alpha}, \quad k = n^{1/2 - \epsilon}, \quad q = k^{-\alpha - \epsilon}$$
Robust versions of planted clique

- $G(n, p, k, q)$ – Erdős–Rényi graph of density $p$ with a randomly planted Erdős–Rényi subgraph of size $k$, density $q$.

**Theorem 3**: Achieving regret $\sqrt{nl^{1-\Theta(\alpha,\epsilon)}}T$ refutes the planted dense subgraph conjecture.

**Planted dense subgraph conjecture**:

Distinguishing between $G(n,p)$ and $G(n,p,k,q)$ with algorithms (Bhaskara et al. 2010) impossible in polynomial time for

\[ p = n^{-\alpha} \quad k = n^{1/2-\epsilon} \quad q = k^{-\alpha-\epsilon} \]
Open problems

- Better regret in sub-exponential time?
- Hardness under more standard assumptions?
- More computational hardness results in online learning?
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Thanks!