Bandit Convex Optimization: \( \sqrt{T} \) Regret in One Dimension

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Bandit Convex Optimization

- **Adversary** picks convex \( f_1, f_2, \ldots, f_T : K \rightarrow [0,1] \) over convex \( K \subseteq \mathbb{R}^n \)
- On rounds \( t=1, \ldots, T \), **player**:
  1. chooses \( x_t \in K \) at random
  2. incurs loss \( f_t(x_t) \)
  3. observes \( f_t(x_t) \) and nothing else

- Player’s goal: minimize regret

\[
R_T = \mathbb{E} \left[ \sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{K}} \sum_{t=1}^{T} f_t(x) \right]
\]
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**Thm:** for $n=1$, there exists algorithm for which $R_T = \tilde{O}(\sqrt{T})$

**Note:** no additional assumptions (not even Lipschitz)
Previous results

- $T^{5/6}$ for general convex [Flaxman, Kalai, McMahan ’05]
- $T^{3/4}$ for Lipschitz [Flaxman, Kalai, McMahan ’05]
- $T^{1/2}$ for linear + Lipschitz [Dani et al ’08, Abernethy et al ’08]
- $T^{2/3}$ for strongly convex + Lipschitz [Agarwal et al ’10]
- $T^{2/3}$ for smooth [Saha & Tewari’ 11]
- $T^{1/2}$ for stochastic i.i.d. + Lipschitz [Agarwal et al ’11]
- $T^{1/2}$ for smooth + strongly convex [Hazan & Levy ’14]

Our result: first tight bound without further assumptions
Main ideas

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4. Bayesian regret analysis via information-theoretic arguments

[Russo & van Roy ’14]
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5. Novel use of convexity: new “**local-to-global**” property of convex functions