Pattern encoding on the Poincaré Sphere

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A graphical tool for pattern encoding
Inspiration: Encoding Polarization States

\[(\frac{E_x}{a_x})^2 + (\frac{E_y}{a_y})^2 - 2\frac{E_x}{a_x}\frac{E_y}{a_y}\cos\delta = (\sin\delta)^2\]

Poincaré sphere  (Henri Poincaré, 1892)

- $S_1 = S_0 \cos(2\chi) \cos(2\psi)$
- $S_2 = S_0 \cos(2\chi) \sin(2\psi)$
- $S_3 = S_0 \sin(2\psi)$
Applications in communications

Optical communications with **POLarization Shift Keying (POLSK)** modulation [S. Benedetto and P. Pogiolini, 1992]

![2-POLSK](image1)
![4-POLSK](image2)
![8-POLSK](image3)


![6mm 125](image4)
![8mm 125](image5)

Examples of 6-D constellations extracted from the $E_6$ lattice [AP, V. Pizurica, V. Šenk, 1998]
Visual patterns on the Poincaré sphere?

Possible coding for visual patterns
Visual patterns on the Poincaré sphere?

Possible coding for visual patterns
Some parallels with polarization encoding

- Texture ↔ Polarization
  - Orientation
    - \( \alpha \)
  - Angle
    - \( \alpha \)
- "Bright-dark balance"
  - Eilpticity
- "Phase"
  - Orientation
Two examples with different formulations of the elevation angle

Formulation 1

Formulation 2
Take an intersection of the sphere with any plane parallel to the equatorial plane.

Any point on the resulting circle is a projection of a 4D line.

⇒ The circle extends to a **scale cylinder**
Take from each scale cylinder a cross section at distance $s_k$ from its base → All the resulting circles make a new sphere for scale $s_k$

The resulting spheres constitute a **scale hypersphere**
Unfolding and packing together the scales

Pool the scale cylinder out $\rightarrow$ unfold the scales in 4th dimension
Make the scale cylinder collapse $\rightarrow$ project 4D space on a 3D space where each point corresponds to a variety of scales
Constructing a toy example

Degree of regularity
 Dominant direction
 Mean intensity
Dominant direction estimation: idea

Consider a zero mean image patch $w = \{w_{i,j}\}$, $w_{i,j} = I_{i,j} - \mu_I$

E.g. $\begin{bmatrix} \square & -1.5 & \square & 1 \end{bmatrix}$

$$\sum_{j} w_{i,j} \quad \sum_{j} |w_{i,j}|$$

$$R_h = \sum_{i} \sum_{j} |w_{i,j}| = 0$$

$$R_v = \sum_{j} \sum_{i} |w_{i,j}| = 30$$
Dominant direction estimation: idea

Consider a zero mean image patch \( w = \{ w_{i,j} \}, \ w_{i,j} = I_{i,j} - \mu_I \)

E.g. \( -1.5 \) \( 1 \)

Design a direction estimation method based on the ratio of \( R_v \) and \( R_h \).
Dominant direction estimation: idea

Consider a zero mean image patch \( w = \{w_{i,j}\}, \ w_{i,j} = I_{i,j} - \mu_I \)

E.g. \(-1.5\) \(1\)

\(R_d1 > R_d2\)

\(R_d2\)

\(R_d1\)

\(R_h = 11\)

\(R_v = 11\)

Two diagonal projectors are sufficient to remove mirroring ambiguity.
Dominant direction estimation: method

\[ w = \{ w_{i,j} \} \] zero-mean image patch

\[ U_{k,l} = \frac{1}{|L_{k,l}|} \sum_{(i,j) \in L_{k,l}} w_{i,j} \]

\[ R_k = \frac{1}{\rho_k} \sum_{k=1}^{\rho_k} |U_{k,l}| \]

number of projection rays for direction \( k \)

Normalization:

\[ r_{h,v} = \frac{R_{h,v}}{(R_{h}^2 + R_{v}^2)^{\frac{1}{2}}} \]

\[ r_{d1,d2} = \frac{R_{d1,d2}}{(R_{d1}^2 + R_{d2}^2)^{\frac{1}{2}}} \]

Absolute value of the sum of elements, normalized by their number
Dominant direction estimation: method

\[ k = v \]

\[ R_v \]

\[ k = h \]

\[ R_h \]

\[ k = d_1 \]

\[ R_{d1} \]

\[ k = d_2 \]

\[ R_{d2} \]

\[ \psi = \arctan \left( \frac{r_v}{r_h} \right) + d_{corr} \cdot 2 \left( \frac{\pi}{2} - \arctan \left( \frac{r_v}{r_h} \right) \right) \]

\[ d_{corr} = \begin{cases} 
    0, & \text{if } r_{d1} \geq r_{d2}, \\
    1, & \text{otherwise.}
\end{cases} \]

Absolute value of the sum of elements, normalized by their number
Dominant direction estimation: examples

![Reference patch](image)

![Graphs](image)

Estimated Angle vs Angle [deg]
Dominant direction estimation: examples

- Reference patch
- Estimated Angle

Graph showing estimated angles for different directions.

### Reference Patch
- 2 rows, 4 columns

### Estimated Angle Graph
- "r_v""r_h""d1""d2"
Dominant direction estimation: examples

reference patch

Estimated Angle

Ang [deg]

Estimated Angle

Ang [deg]
Dominant direction estimation: examples
Dominant direction estimation: examples

- Reference patch
- Degraded patch

Graphs showing the estimated angles and the performance of different metrics. The graphs illustrate the variation of estimated angles with respect to the actual angle.
Dominant direction estimation: examples

Estimated Angle vs Angle [deg]

Reference patch

Degraded patch
Encoding the level of grey

Let $T$ denote a normalized mean intensity of an image patch $I = \{I_{i,j}\}$.

$$T = \frac{\sum_{i=1}^{M} \sum_{j=1}^{N} I_{i,j}}{255MN}, \quad 0 \leq T \leq 1,$$

and define

$$\Theta = 2\chi = (T - 0.5)\pi$$
Encoding patch regularity

$E_I = - \sum_j p_j \log_2(p_j)$

with 2 levels: $\max\{E_I\}=1$; with 256 levels: $\max\{E_I\}=8$

$\rho \in [0, 1]$

$\rho_E = \min\left(1 - \frac{E_I - 1}{7}, 1\right)$
Encoding patch regularity

Think of the degree of regularity as the degree of orientedness and examine local directional consistency (LDC).

Let $\psi_i$ denote dominant orientation of a sub-block $i$ and let $h_\psi$ denote the histogram of $\psi=\{\psi_i \ldots \psi_i\}$.

$$\rho_{LDC} = \frac{B - b}{B - 1}$$

$B$ – total number of bins in $h_\psi$

$b$ – number of populated bins

(with counts above a small threshold)
Patch encoding example

with

$\rho_{LDC}$

with

$\rho_E$
Some possible applications

• Patch clustering
• Analyzing learned dictionaries of image atoms
• Generating dictionaries of image atoms
Applications: Patch clustering

Random patches of size 16x16 taken from four image regions highlighted with the corresponding colors.
Applications: Dictionary analysis

Examples of multiscale dictionaries from [Mairal, Sapiro and Elad, 2008]
Dictionary analysis: Zoom In

DB1, s=0 [Mairal, Sapiro and Elad, 2008]

Notice lack of diagonal highly oriented atoms – this is visible in the Poincaré representation!
Dictionary analysis: Zoom In

DB1, s=3 [Mairal, Sapiro and Elad, 2008]

Notice many diagonal atoms – reflected in the Poincaré code
Applications: Encoding image atoms
Applications: Encoding image atoms
Applications: Encoding image atoms

Sky, Ridders, KSVD 16x16

People, Ridders, KSVD 16x16
Generating dictionaries of image atoms

Extract from the three Stokes parameters the regularity, direction and elevation (mean grey tone) and generate randomly the corresponding patterns.

Examples with atom size 8x8
Generating dictionaries of image atoms

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Extract from the three Stokes parameters the regularity, direction and elevation (mean grey tone) and generate randomly the corresponding patterns:

- Examples with atom size 8x8
- Examples with atom size 16x16
Generating dictionaries of image atoms

Extract from the three Stokes parameters the regularity, direction and elevation (mean grey tone) and generate randomly the corresponding patterns.

Examples with atom size 8x8

Examples with atom size 16x16
Generating dictionaries of image atoms

Extract from the three Stokes parameters the regularity, direction and elevation (mean grey tone) and generate randomly the corresponding patterns

Examples with atom size 8x8

Examples with atom size 16x16
Generating dictionaries of image atoms

72 points

3x3
Generating dictionaries of image atoms

72 points

6x6
Generating dictionaries of image atoms

72 points

8x8
Generating dictionaries of image atoms

256 points

$2D_{256}(8 \times 8)$
Image reconstruction examples

In all reconstructions:

atom size: 8x8; sparsity: 5;
reconstruction method: OMP

PSNR = 31.62 dB

72 points
Image reconstruction examples

In all reconstructions:

atom size: 8x8; sparsity: 5;
reconstruction method: OMP

PSNR = 33.63 dB
Image reconstruction examples

In all reconstructions:
- atom size: 8x8; sparsity: 5;
- reconstruction method: OMP

PSNR = 34.33 dB

256 points
2 random dictionaries (2x256)
Reconstruction performance

atom size: 8x8; sparsity: 5; reconstruction method: OMP
Sphere packings


Tables of Spherical Codes with Icosahedral Symmetry
R. H. Hardin, N. J. A. Sloane and W. D. Smith
Example dictionary from a spherical code
Reconstruction performance

atom size: 8x8; sparsity: 5; reconstruction method: OMP
Summary

- A graphical tool was presented for encoding visual patterns
- Possible applications include
  - Patch clustering
  - Visualizing properties of learned dictionaries of image atoms
  - Generating dictionaries of image atoms
References

Material from this presentation:

Spherical codes, packings, lattice coding

POLSK systems with spherical codes

Learning dictionaries of image atoms