# Learning Multifractal Structure in Large Networks

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# Setting

We want a simple, scalable method to model networks and generate random (undirected) graphs

- Looking for random graph generators that can mimic real world graph structure
  - Power law degree distribution,
  - High clustering coefficient, etc.
- Many models have been proposed, starting with Erdos-Renyi graphs
- Relatively recent models: SKG [Leskovec et al. 2010], BTER [Seshadhri et al. 2012], TCL [Pfeiffer et al. 2012]
- In 2011 Palla et al. introduce multifractal network generators, 'generalizing' SKG

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We propose methods to make MFNG a feasible framework to model large networks

- First, we give an intuitive theoretical result that opens the door to scalable estimation
- We show how we can fit MFNG to graphs using mehod of moments estimation, with runtime independent of the size of the graph
- We develop a fast heuristic for sampling MFNG
- We demonstrate the effictiveness of our approach in synthetic and real world settings.

# An introduction to Multifractal Network Generators (MFNG)

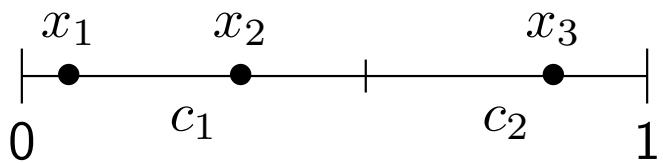
Ingredients

- Number of nodes: $n$
- Number of categories: $m$ with specified lenghts $l_i$
- Number of recursive levels: $k \approx \log_m(n)$
- Probabilities of edges between nodes, based on categories, stored in matrix $P \in [0,1]^{m \times m}$

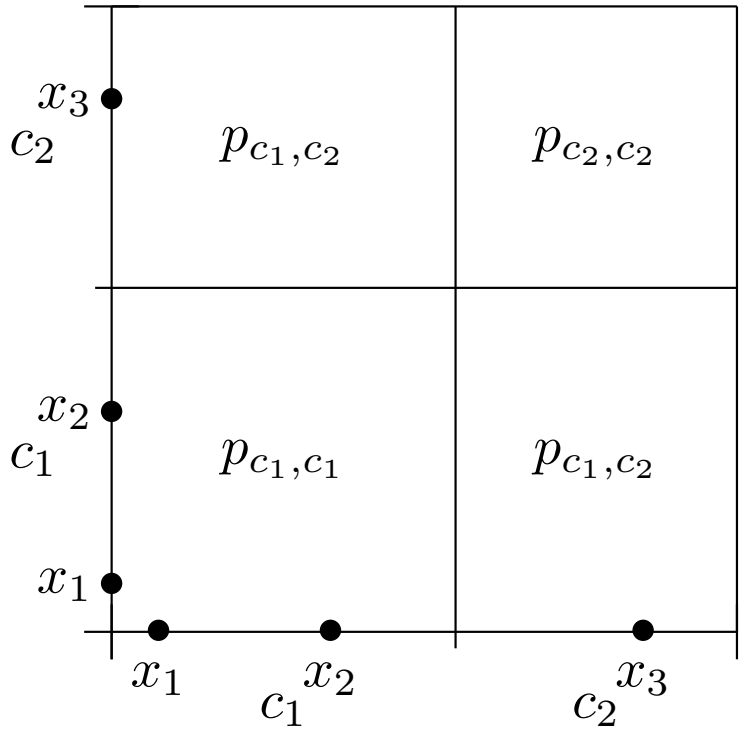
# Generating a graph with no recursion

Let's consider the simple case first: $k = 1$

- Begin with a line: $[0, 1]$
- Divide the line in $m$ intervals (or categories) with lengths $l_1, l_2, \ldots, l_m$
- Sample nodes on the line according to a uniform distribution: this gives every node a category



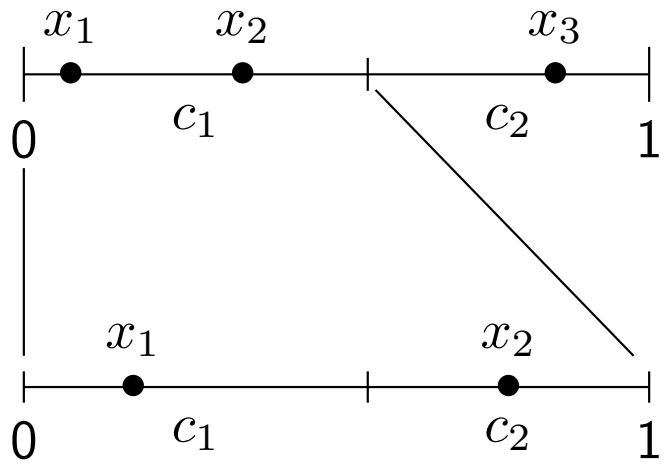
# From line to square




- For any two nodes $u \in c_i, v \in c_j$, add an edge with probability according to $p_{c_i,c_j}$

# Adding recursion

For subsequent levels, we subdivide the intervals again to find categories of nodes in the next layer



# Adding recursion

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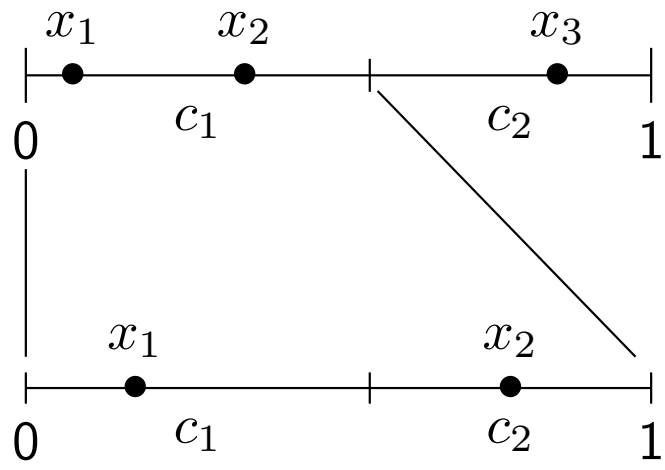



Now add an edge between nodes by multiplying probabilities corresponding to categories in each layer.
In the above two layer example

- node $x_1$ has categories $(c_1, c_1)$,
- node $x_2$ has categories $(c_1, c_2)$, and
- node $x_3$ has categories $(c_2, c_2)$

And hence, we add edge $(x_1, x_2)$ with probability $p_{c_1,c_1} p_{c_1,c_2}$.

# Expanding the recursion

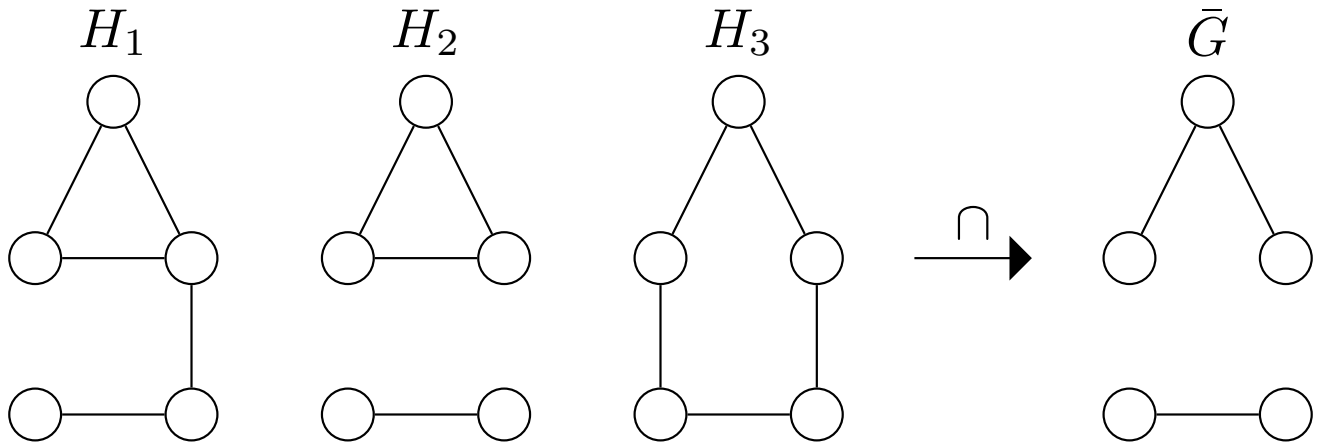
So we can get a full probabilistic adjacency matrix $Q \in [0,1]^{m^k \times m^k}$ by expanding all recursive levels

Problem: $Q$ grows fast with $k$. Difficult to do inference.

Intuitively, we should not have to do this.

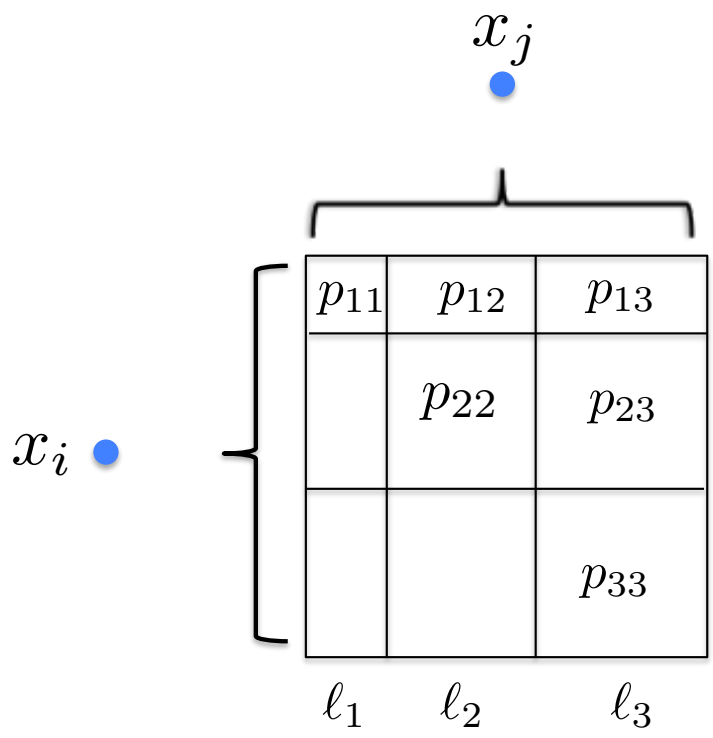
# Main theoretical result

Consider sampling $k$ graphs from a MFNG with $1$ recursive level, and construct a new graph $\bar{G}$ by taking the intersection over graphs:




Then $\bar{G}$ has same distribution as a graph $G$ generated from a MFNG with $k$ recursive levels.

# Computing expected number of edges is easy



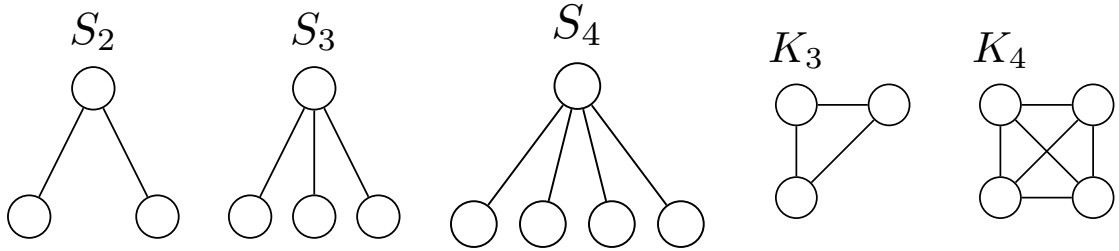

$$\mathbf{Prob}((u,v) \in E) = p = \sum_{i=1}^{3}\sum_{j=1}^{3} \ell_i \ell_j p_{ij}, \quad \mathbb{E}\{|E|\} = \binom{n}{2} p^k$$

So computing $p$ is $O(m^2)$ instead of $O(m^{2k})$.

# Computing moments of certain subgraphs is easy

With above theory, we can easily compute the expected number of...

- edges, wedges, $3$-stars, $4$-stars ...
- triangles, $4$-cliques...



# We can learn multifractal structure quickly

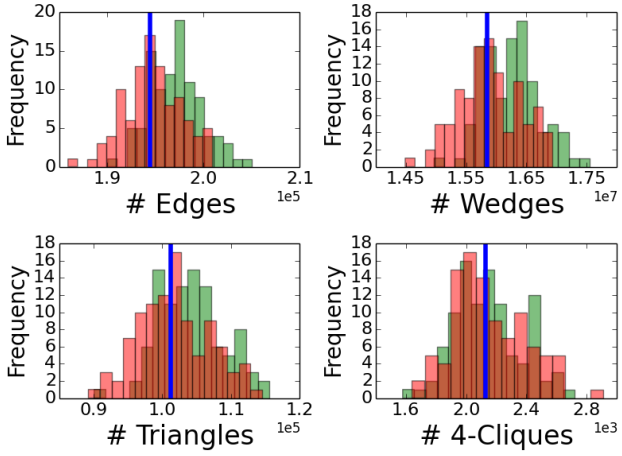
Method of moments:

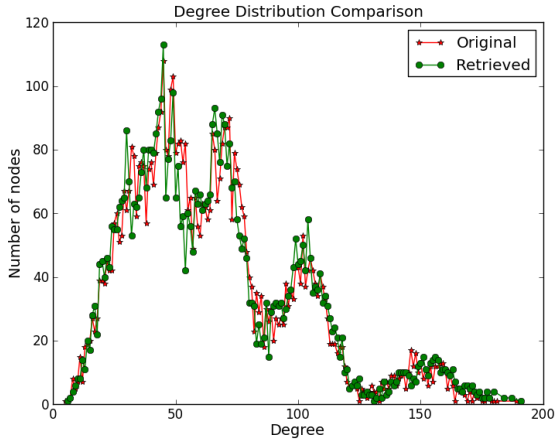
1. Count number of wedges, 3-stars, triangles, 4-cliques, etc. in network of interest
2. Try to find parameters such that the expected values, $\mathbb{E}[F_i]$, match the empirical counts, $f_i$

$$\begin{aligned}
\underset{P,\ell,r}{\text{minimize}} \quad & \sum_i \frac{|f_i - \mathbb{E}[F_i]|}{f_i} \\
\text{subject to} \quad & 0 \le p_{ij} = p_{ji} \le 1, \quad 1 \le i \le j \le c \\
& 0 \le \ell_i \le 1, \qquad\qquad 1 \le i \le c \\
& \sum_{i=1}^{m} \ell_i = 1
\end{aligned}$$

Key idea: Once we have the counts $(f_i)$, this optimization routine is independent of the size of the graph.

# Method of moments recovers small synthetic graphs

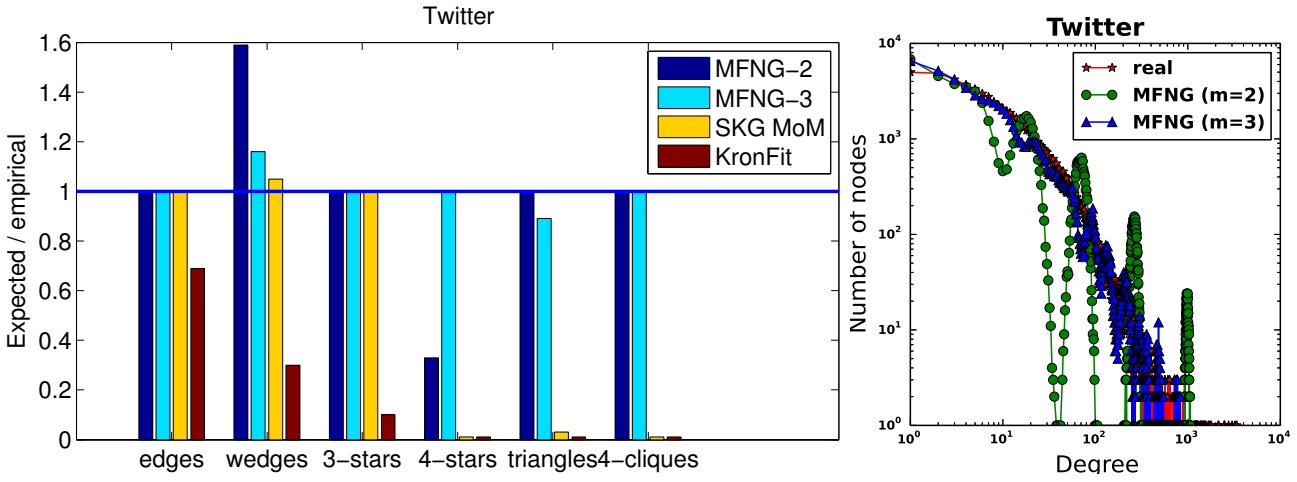



Original MFNG, Single sample, Recovered MFNG

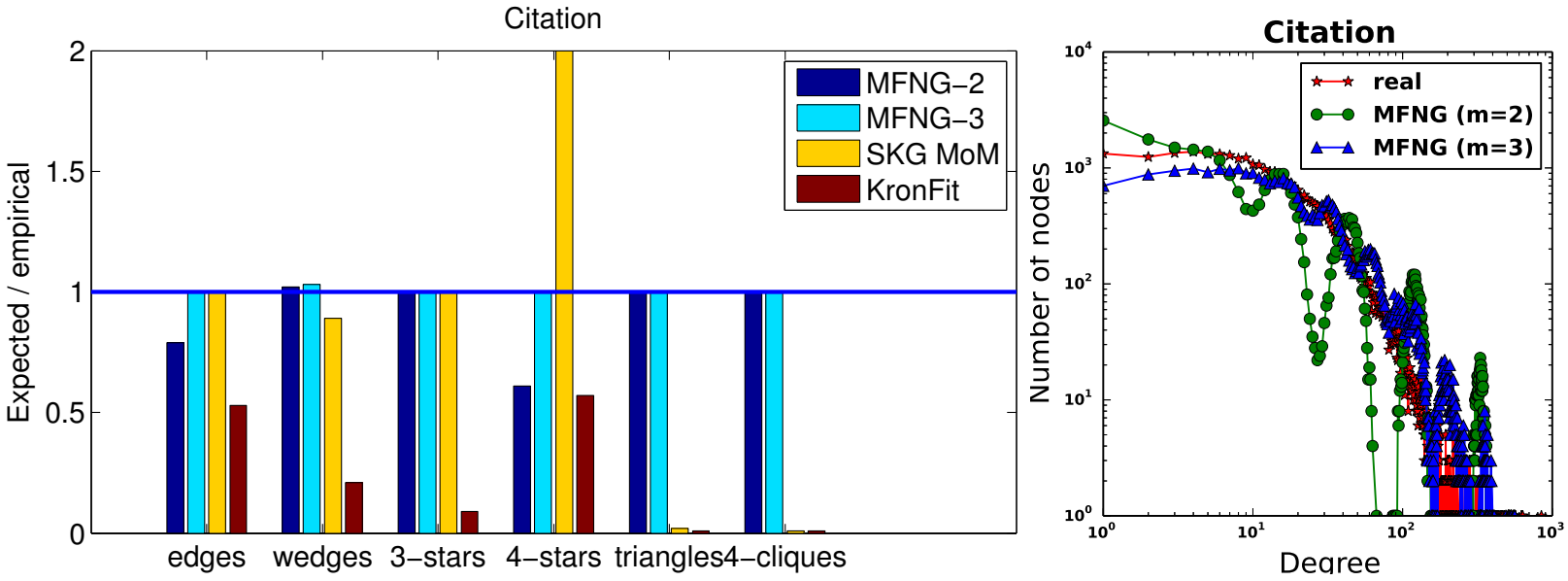
| | $\lvert V\rvert$ | $m$ | $k$ | $\ell_1$ | $\ell_2$ | $p_{11}$ | $p_{12}$ | $p_{22}$ |
|---|---|---|---|---|---|---|---|---|
| Original | 6,000 | 2 | 10 | 0.25 | 0.75 | 0.59 | 0.43 | 0.78 |
| Recovered | 6,000 | 2 | 9 | 0.2728 | 0.7272 | 0.5431 | 0.4101 | 0.7593 |

# Twitter network




Comparison against a method of moments for Stochastic Kronecker Graphs [Gleich and Owen 2012] and KronFit [Leskovec et al. 2010]

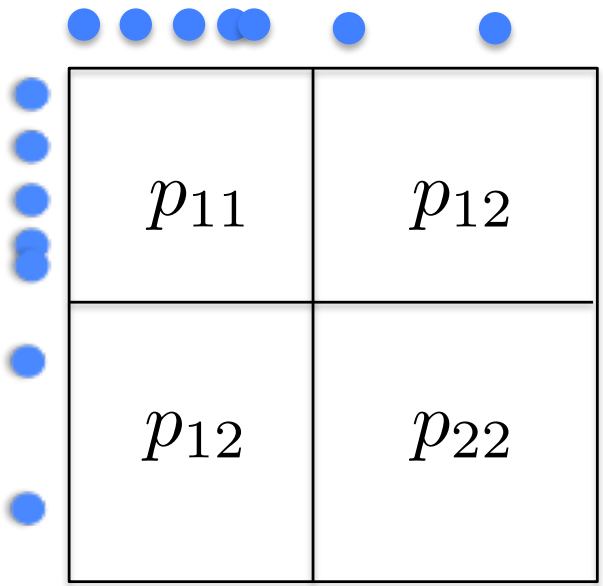
# Citation network




- Triangles and $4$-cliques are again matched in expectation.
- We employ "noisy SKG" strategy [Seshadhri et al. 2013] to dampen the degree distribution.

# Fast sampling is challenging

- Naive way: flip a coin for each $O(n^2)$, while we would like $O(|E|)$
- Idea from SKG: fix number of edges and then use 'ball-dropping'
- Problem for MFNG: many nodes can fall into a single box, we have to ensure we still sample enough edges from that box



# Conclusion

We proposed methods to make MFNG a feasible framework to model large networks

- First, we give an intuitive theoretical result that opens the door to scalable estimation
- We show how we can fit MFNG parameters to arbitrarily largh graphs using mehod of moments estimation
- We develop a fast heuristic for sampling MFNG
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Questions?


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