Volumetric Ellipsoids: An exploration basis for learning

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COLT' 14

Experiment Design

n patients, each with d features: $x_1,\ldots,x_n$ in R^d

experiment = choose x_i , gives noisy measurement:

$$\langle w^*, x_i \rangle + \epsilon_i$$

Goal: regress on data with as few measurements as possible (learn w^*)

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Our objective: Find $S \subseteq \{x_1,\ldots,x_n\}$ of minimal size with

$\max_x \mathrm{Var}[\langle x, w-w^* \rangle] \leq \epsilon$

Low variance exploration basis

Given K (discrete or continuous):

- Subset of K: $S = \{v_1,\ldots,v_t\} \subseteq K$
- Small cardinality – t
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<u>Barycentric spanner</u>

$\max_i |\alpha_i| \leq 1$

[Awerbuch Kleinberg]

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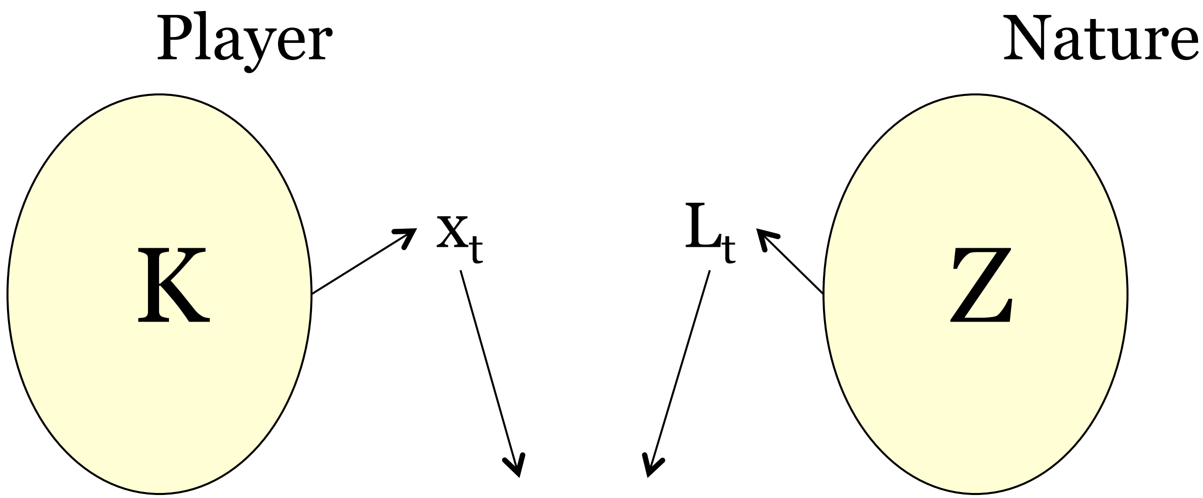
$$\langle x, w^* \rangle = \sum_i \alpha_i \langle v_i, w^* \rangle$$

$$\mathrm{Var}[\langle x, w^* \rangle] = \mathrm{Var}\left[\sum_i \alpha_i \langle v_i, w^* \rangle\right] \leq \|\alpha\|^2 \sigma^2$$

Application: BLO



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- Round: player picks x_t, nature picks L_t, loss = ℓ_t

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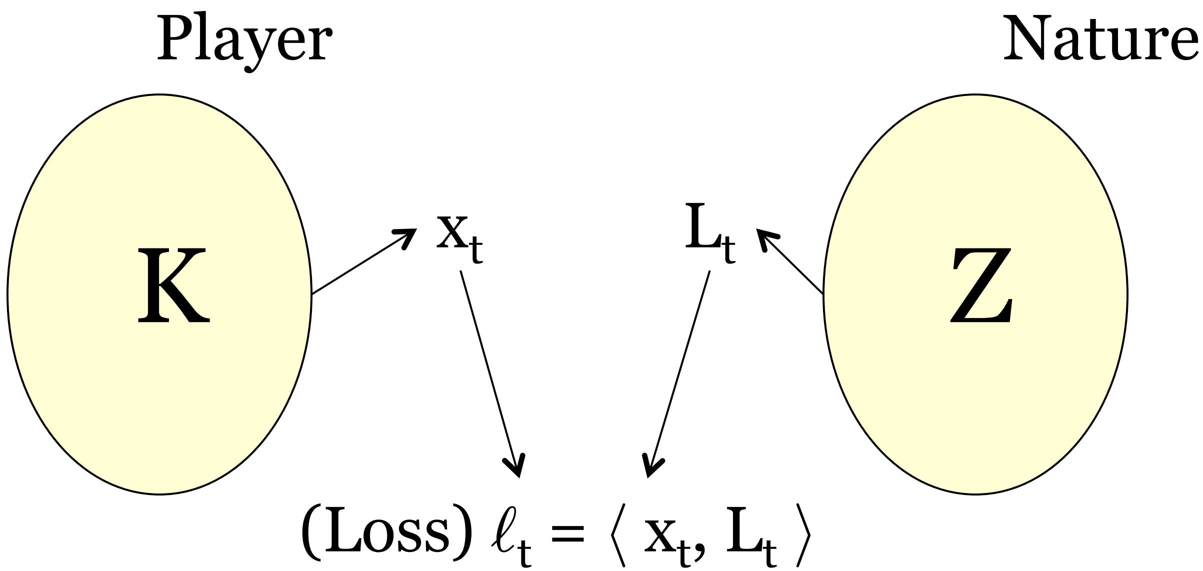

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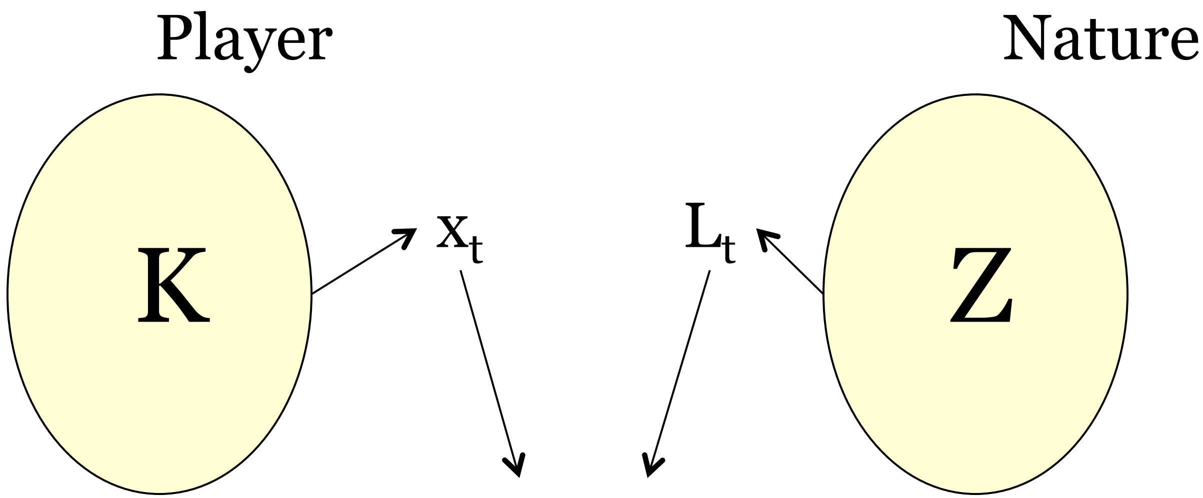

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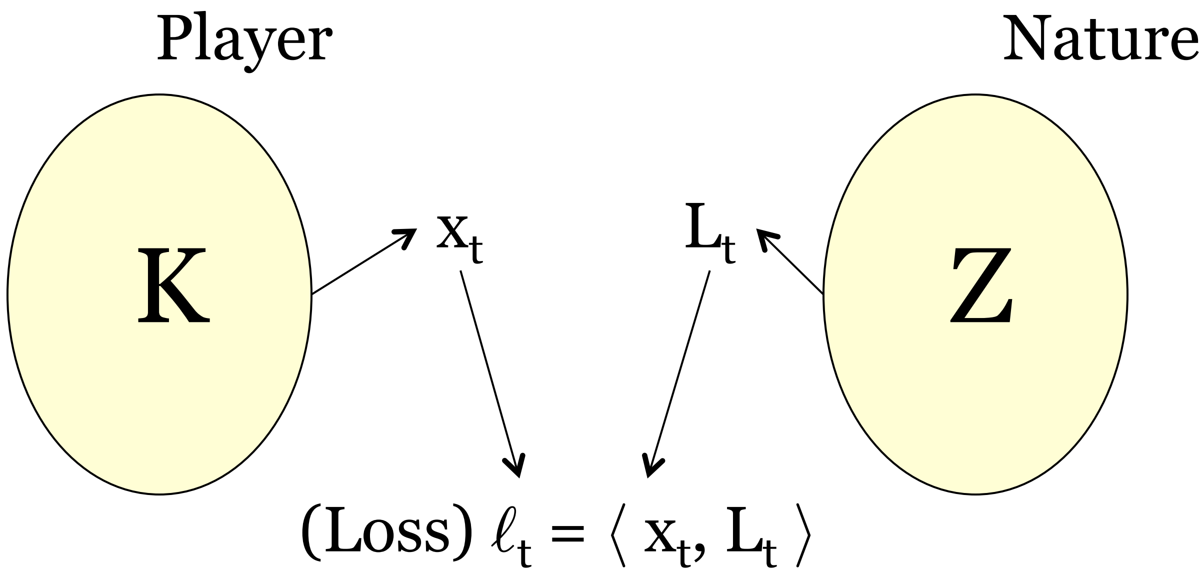

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- Regret = $\sum_t \langle x_t, L_t \rangle - \sum_t \langle x^*, L_t \rangle$

BLO Results

Paper	Regret	Effici-ent?	Comments	Method
[Awerbuch Kleinberg]	$T^{2/3}$ poly(d)	Yes		Barycentric spanner
[Abernethy Hazan Rakhlin]	$d(\theta T)^{1/2}$	Yes	θ =self concordant barrier = O(d), but can be O(1)	Self concordant barriers
[Bubeck, Cesa-Bianchi, Kakade]	$dT^{1/2}$	No	matches lower bound (for general convex bodies)	John's decomposition
Here	$dT^{1/2}$	Yes	For some specific bodies, better regret is possible. Worst case is $dT^{1/2}$	Volumetric Spanners

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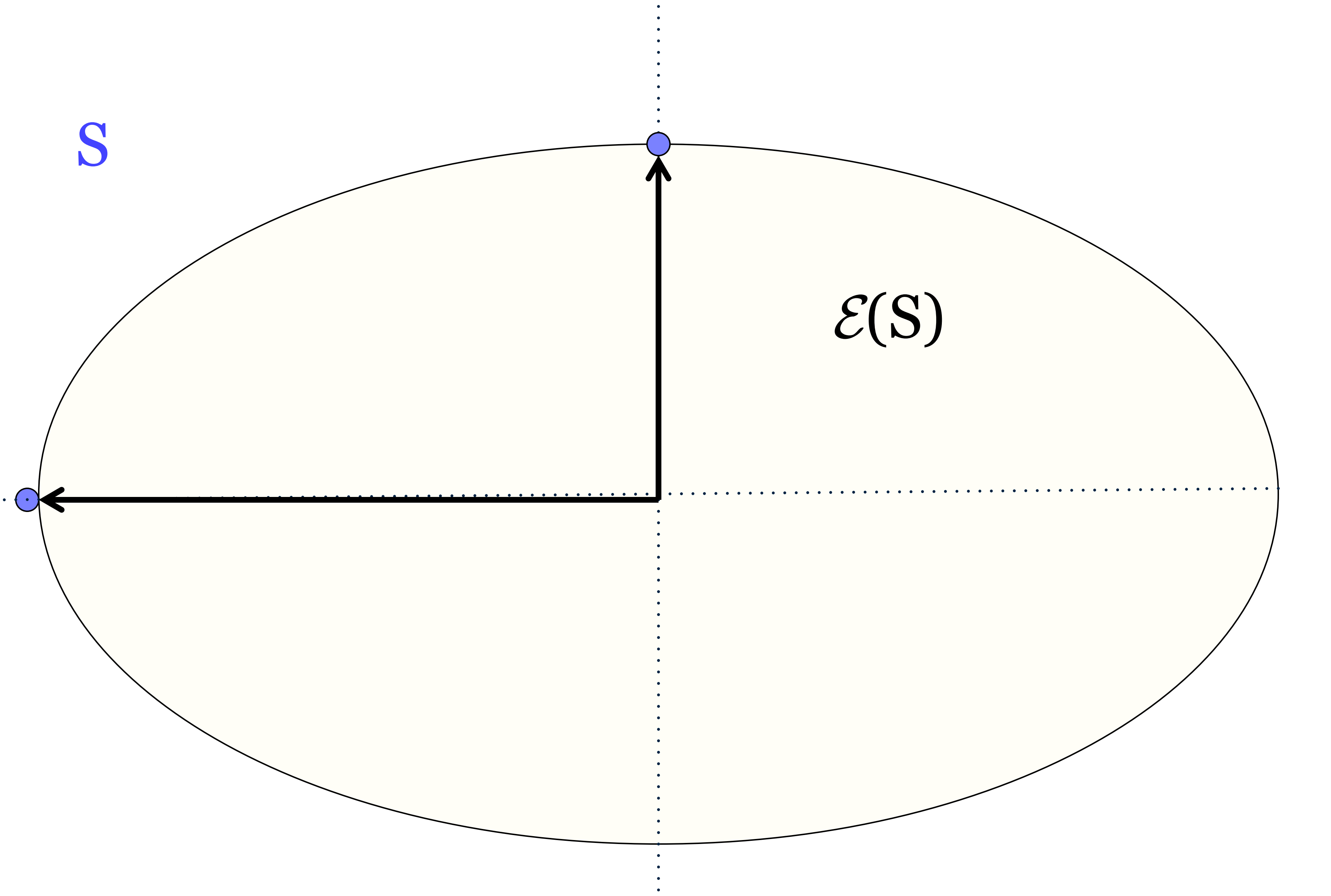
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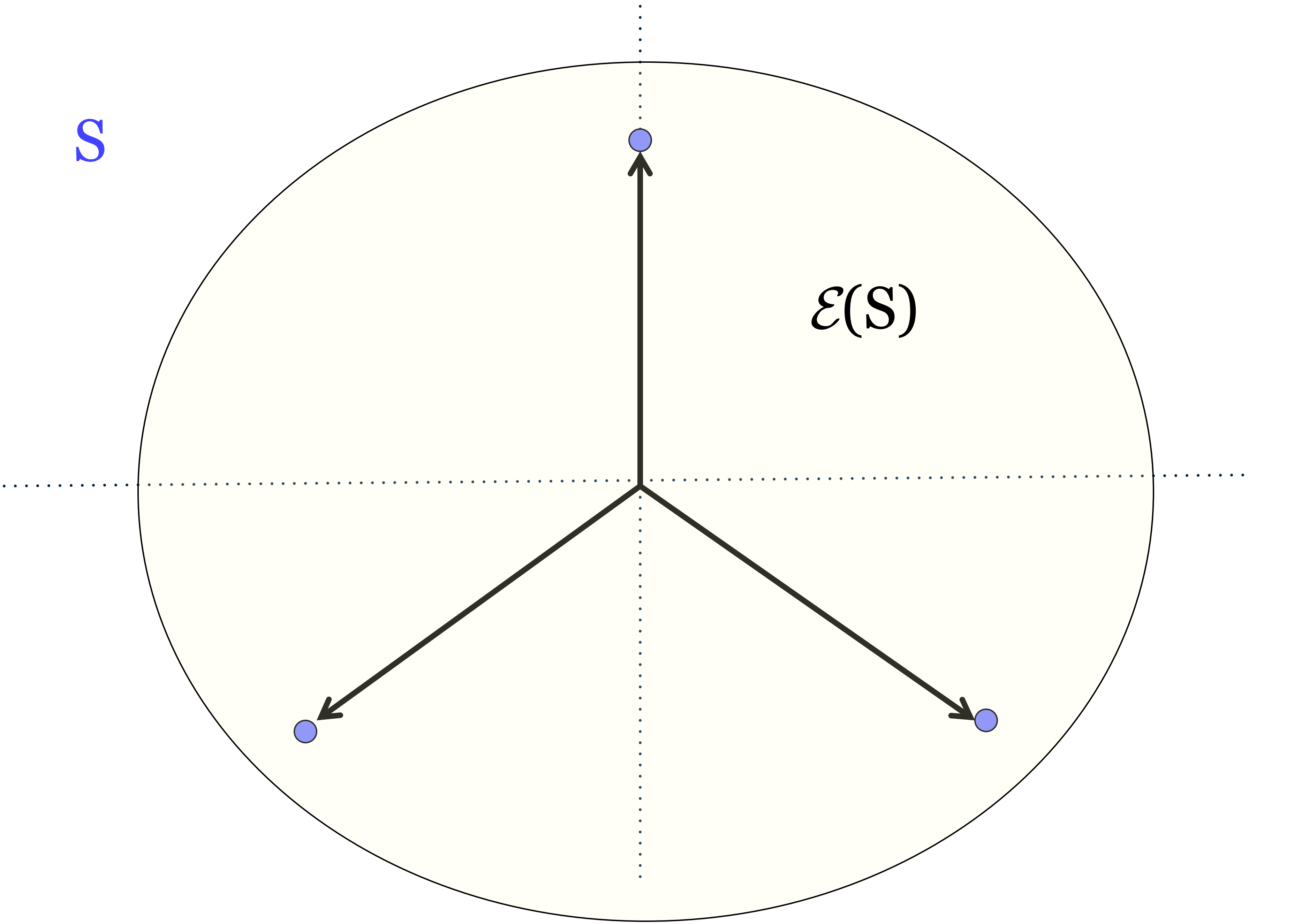
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Objective: find min $|S|$ s.t. $S \subseteq K \subseteq \mathcal{E}(S)$

Questions

- Given K, how small is |S| of volumetric ellipsoid? "order(K)"
- Can we find it efficiently?
- What bounds does it give for experiment design?
- Can it be used in other learning problems?

What is order(K) ?

Thm: $d < \text{order}(K) \leq 12d$

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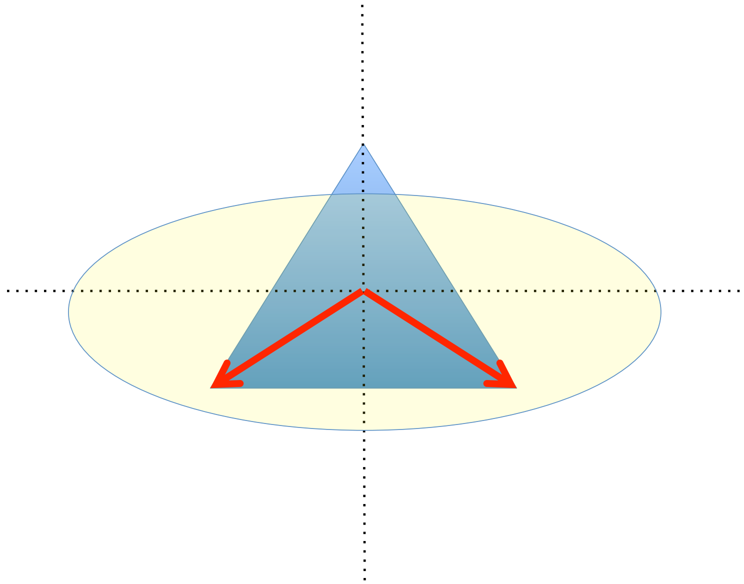
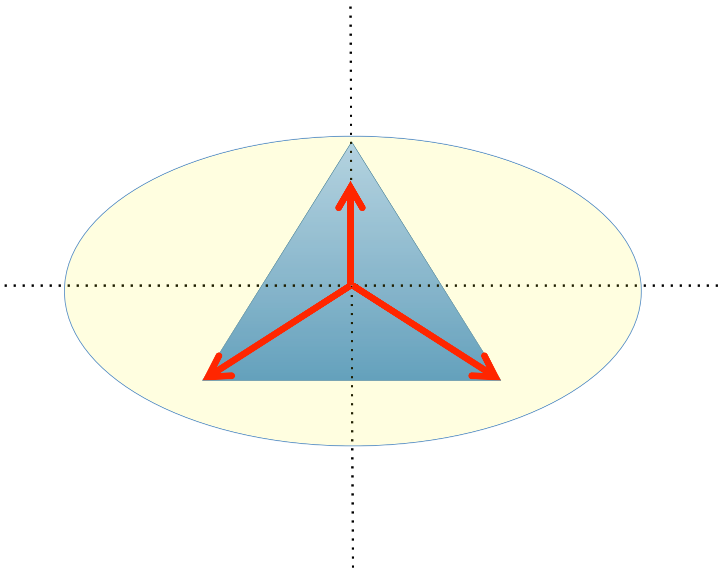
Order(K) > d





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Proof:

John's decomposition provide a "broken solution"; can be seen as S of size d^2. Twice-Ramanujan sparsifiers (BSS' 12) provide method to reduce to 12d.

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Application in experiment design:

Test at most $O(d / \epsilon^2)$ patients to regress up to precision ϵ over all patients!

Constructive Results for Finite K

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- Lem: For large |K|, we give a simpler $|K|d^2$ construction giving size O(d log(d) log(|K|))

Constructive Results for Convex K

Two types of approximation:

1) Ratio-Spanner: $K \subseteq \rho \cdot \mathcal{E}(S)$
2) Exp-Spanner: for $x \in K$, $\Pr[x \in \theta \cdot \mathcal{E}(S)] \geq 1 - \epsilon^{\theta}$

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Application to Bandit Linear Optimization: First efficient and optimal-regret* algorithm

- * convex sets, L2 assumption, can do better for some specific convex bodies
- Geometric hedge framework: Use approximate volumetric spanner as exploration basis

Conclusion

- Experiment design: test at most $O(d / \epsilon^2)$ patients to regress up to precision ϵ over all patients!

- Bandit Linear Optimization: first efficient and optimal-regret algorithm for general convex sets

- Active learning applications?

Thank you!