Computational Limits for Matrix Completion

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What is Matrix Completion?

Matrix Completion:

- Nature has some rank-k matrix M, ($k << n$)

$$M = \begin{bmatrix} 1 & 1 & 3 & 2 & 2 \\ 2 & 2 & 6 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 3 & 3 \\ 3 & 3 & 7 & 5 & 5 \end{bmatrix}$$

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A Geometric Viewpoint

- Low-rank completions have factorizations $M' = UV^T$

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Goal: Understand when Matrix Completion is tractable

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Natural algorithms, even when the number of revealed entries is very small (for example see [Candés and Recht '09], [Candés and Tao '10], and [Recht '11]), but only under two assumptions:

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Are these assumptions required for tractability?

Previous Hardness Results

- [Peeters '96]: Rank-k completion is NP-hard for $k \geq 3$.
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These results say nothing about approximation hardness

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Does Matrix Completion remain hard even under these relaxations?

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- Even if M is rank k, hard to find any completion M' of rank $r > k$ approximating P.

To our knowledge, these are among the first approximation hardness results for Matrix Completion.

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Despite tremendous effort for a long period of time (see for example [Wig82], [Blum89], [BK97],[KMS98], [ACC06],[Chl07], [KT12]) best algorithms for $k = 3$ are $r = O(n^{0.2038})$.

Main Theorems

Theorem 1

Assume (k, r)-Graph Coloring is intractable for any constants k, r. Then for c, k, r constants, there is no poly-time algorithm that, given a partial matrix P with a completion M satisfying:

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- **Good Rank Approximation**: $\text{rank}(M') \leq r$.
- **Good Entry Approximation**: For $\epsilon < 1/(2cr)^2$,

$$\sum_{\substack{\text{revealed} \\ \text{entries}}} (M'_{ij} - P_{ij})^2 \leq \epsilon \sum_{\substack{\text{revealed} \\ \text{entries}}} P_{ij}^2$$

Proof of Theorem 1

Recall a low-rank completion is a set of low-dimensional vectors satisfying dot product constraints:

$$\begin{bmatrix} & | & \\ & v_3 & \\ & | & \end{bmatrix}$$

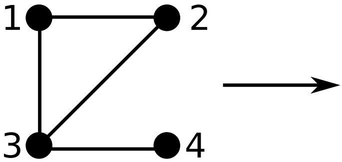
$$\begin{bmatrix} \\ — \quad u_3 \quad — \\ \\ \end{bmatrix} \begin{bmatrix} 1 & 1 & ? & ? & 2 \\ 2 & ? & 6 & ? & 4 \\ 1 & ? & \boxed{1} & 1 & ? \\ 2 & 0 & 4 & 3 & ? \\ ? & 1 & 7 & ? & 5 \end{bmatrix}$$

Our goal is to set up the constraints so that any low-dimensional solution solves an NP-hard problem.

Proof of Theorem 1

Reduce from the k-coloring problem on a graph $G = (V, E)$.
Define a $|V| \times |V|$ matrix P_G of dot product constraints:

$$P_G(i,j) = u_i \cdot v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } (i,j) \in E \\ ? & \text{otherwise} \end{cases}$$



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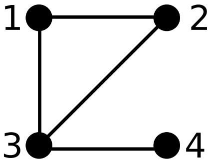

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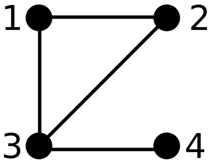

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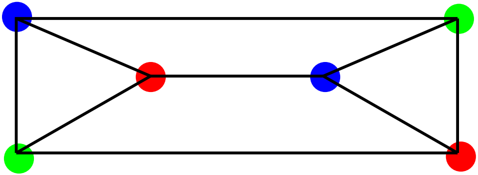
- k-coloring of $G \Rightarrow$ good completion of P_G.
- Good completion of $P_G \Rightarrow$ independent set in G.

Coloring $\Rightarrow$ Good Completion

If G is k-colorable, pick $u_i = v_i = e_{\text{color}(i)}$ a basis vector.



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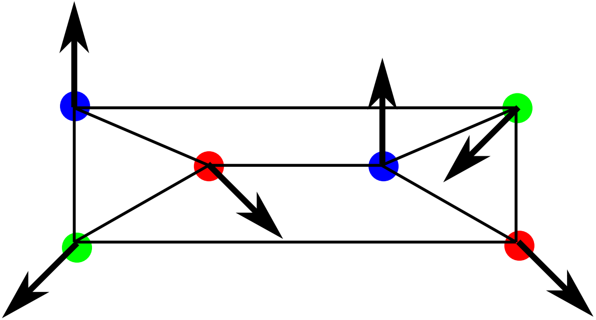

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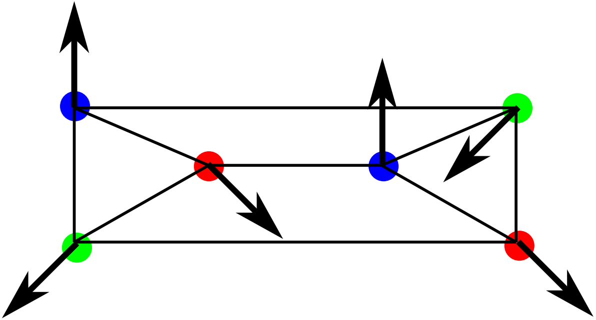


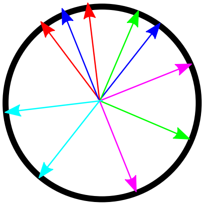
Coherence $\propto \max_i 1/|\text{color}(i)|$

Good Completion ⇒ Independent Set

Given (u_i, v_i) with

- $u_i \cdot v_i = 1$
- $u_i \cdot v_j = 0$ for $(i, j) \in E$

Find a large independent set in G.



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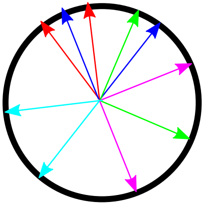

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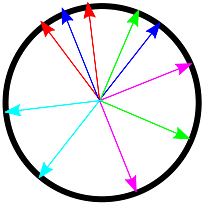

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Instead, look for a direction close to many vector pairs (u_i, v_i). These i will form an independent set.

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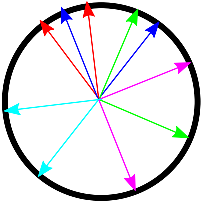

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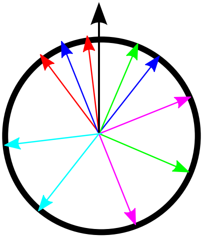

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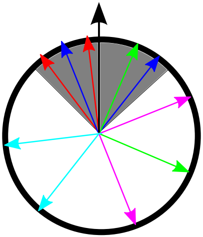
Good Completion $\Rightarrow$ Independent Set

Pick a *random* direction w in $\mathbb{R}^r$, if both (u_i, v_i) lie in a $\pi/2$-cone around w, put $i \in S$.



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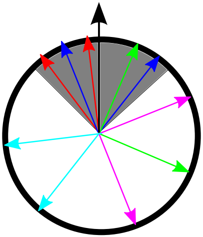

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Two things to show:

- S is an independent set in G
- $|S|$ is $\Omega(|V|)$.

Conclusion

This work has established some of the first approximation hardness results for Matrix Completion:

- If entries are revealed adversarially, Matrix Completion remains hard even when there is an incoherent completion.
- Even when there is a low-rank completion, it is still hard to find merely a rank-approximate or entry-approximate completion.

Further Work

Still a huge amount of work to be done and so much unknown!

- Is randomness alone sufficient for tractability, or is completion hard when the entries are chosen randomly, but the underlying matrix is *not* incoherent?
- We have given hardness for completion with any pair of constants $r \geq k$. What is the largest r such that the problem remains hard?
- We know that coloring is easy for $k = 3$ and $r = O(n^{0.2038})$. Is completion easy for these parameters?